1 ☐ FUNCTIONS AND MODELS

1.1 Four Ways to Represent a Function

In exercises requiring estimations or approximations, your answers may vary slightly from the answers given here.

1. (a) The point $(-1, -2)$ is on the graph of f, so $f(-1) = -2$.

(b) When $x = 2$, y is about 2.8, so $f(2) \approx 2.8$.

(c) $f(x) = 2$ is equivalent to $y = 2$. When $y = 2$, we have $x = -3$ and $x = 1$.

(d) Reasonable estimates for x when $y = 0$ are $x = -2.5$ and $x = 0.3$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-3 \leq x \leq 3$, or $[-3, 3]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2 \leq y \leq 3$, or $[-2, 3]$.

(f) As x increases from -1 to 3, y increases from -2 to 3. Thus, f is increasing on the interval $[-1, 3]$.

2. (a) The point $(-4, -2)$ is on the graph of f, so $f(-4) = -2$. The point $(3, 4)$ is on the graph of g, so $g(3) = 4$.

(b) We are looking for the values of x for which the y-values are equal. The y-values for f and g are equal at the points $(-2, 1)$ and $(2, 2)$, so the desired values of x are -2 and 2.

(c) $f(x) = -1$ is equivalent to $y = -1$. When $y = -1$, we have $x = -3$ and $x = 4$.

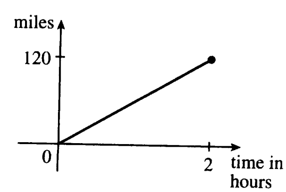
(d) As x increases from 0 to 4, y decreases from 3 to -1. Thus, f is decreasing on the interval $[0, 4]$.

(e) The domain of f consists of all x-values on the graph of f. For this function, the domain is $-4 \leq x \leq 4$, or $[-4, 4]$. The range of f consists of all y-values on the graph of f. For this function, the range is $-2 \leq y \leq 3$, or $[-2, 3]$.

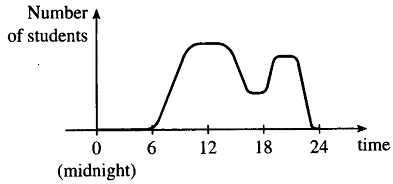
(f) The domain of g is $[-4, 3]$ and the range is $[0.5, 4]$.

3. From Figure 1 in the text, the lowest point occurs at about $(t, a) = (12, -85)$. The highest point occurs at about $(17, 115)$. Thus, the range of the vertical ground acceleration is $-85 \leq a \leq 115$. In Figure 11, the range of the north-south acceleration is approximately $-325 \leq a \leq 485$. In Figure 12, the range of the east-west acceleration is approximately $-210 \leq a \leq 200$.

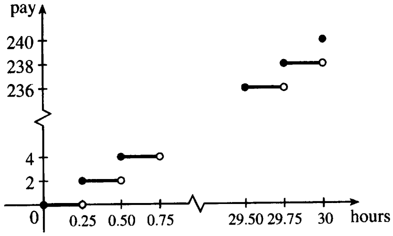
4. *Example 1:* A car is driven at 60 mi/h for 2 hours. The distance d traveled by the car is a function of the time t. The domain of the function is $\{t \mid 0 \leq t \leq 2\}$, where t is measured in hours. The range of the function is $\{d \mid 0 \leq d \leq 120\}$, where d is measured in miles.



Example 2: At a certain university, the number of students N on campus at any time on a particular day is a function of the time t after midnight. The domain of the function is $\{t \mid 0 \leq t \leq 24\}$, where t is measured in hours. The range of the function is $\{N \mid 0 \leq N \leq k\}$, where N is an integer and k is the largest number of students on campus at once.



Example 3: A certain employee is paid \$8.00 per hour and works a maximum of 30 hours per week. The number of hours worked is rounded down to the nearest quarter of an hour. This employee's gross weekly pay P is a function of the number of hours worked h. The domain of the function is $[0, 30]$ and the range of the function is $\{0, 2.00, 4.00, \ldots, 238.00, 240.00\}$.



5. No, the curve is not the graph of a function because a vertical line intersects the curve more than once. Hence, the curve fails the Vertical Line Test.

6. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-2, 2]$ and the range is $[-1, 2]$.

7. Yes, the curve is the graph of a function because it passes the Vertical Line Test. The domain is $[-3, 2]$ and the range is $[-3, -2) \cup [-1, 3]$.

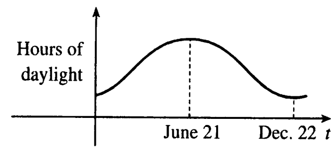
8. No, the curve is not the graph of a function since for $x = 0, \pm1$, and ±2, there are infinitely many points on the curve.

9. The person's weight increased to about 160 pounds at age 20 and stayed fairly steady for 10 years. The person's weight dropped to about 120 pounds for the next 5 years, then increased rapidly to about 170 pounds. The next 30 years saw a gradual increase to 190 pounds. Possible reasons for the drop in weight at 30 years of age: diet, exercise, health problems.

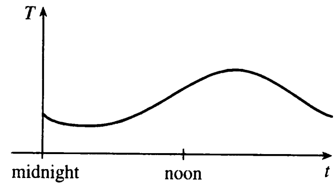
10. The salesman travels away from home from 8 to 9 A.M. and is then stationary until 10:00. The salesman travels farther away from 10 until noon. There is no change in his distance from home until 1:00, at which time the distance from home decreases until 3:00. Then the distance starts increasing again, reaching the maximum distance away from home at 5:00. There is no change from 5 until 6, and then the distance decreases rapidly until 7:00 P.M., at which time the salesman reaches home.

11. The water will cool down almost to freezing as the ice melts. Then, when the ice has melted, the water will slowly warm up to room temperature.

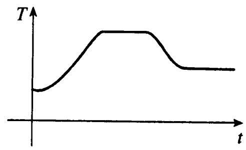
12. The summer solstice (the longest day of the year) is around June 21, and the winter solstice (the shortest day) is around December 22.



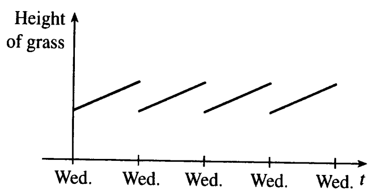
13. Of course, this graph depends strongly on the geographical location!



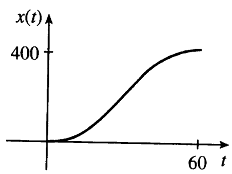
14. The temperature of the pie would increase rapidly, level off to oven temperature, decrease rapidly, and then level off to room temperature.



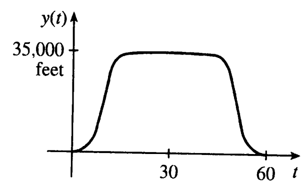
15.



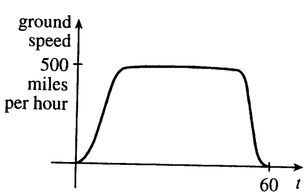
16. (a)



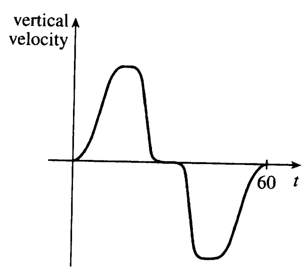
(b)



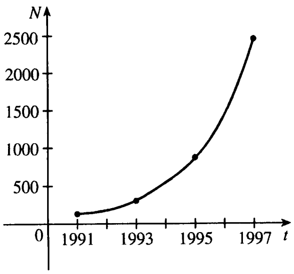
(c)



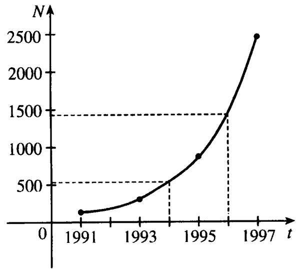
(d)



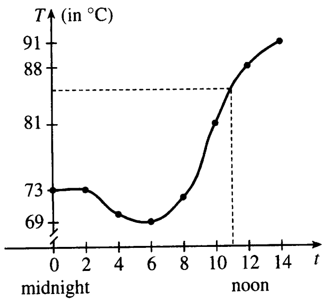
17. (a)



(b) From the graph, we estimate the number of cell-phone subscribers in Malaysia to be about 540 in 1994 and 1450 in 1996.



18. (a)



(b) From the graph in part (a), we estimate the temperature at 11:00 A.M. to be about 84.5°C.

19. $f(x) = 3x^2 - x + 2.$

$f(2) = 3(2)^2 - 2 + 2 = 12 - 2 + 2 = 12.$

$f(-2) = 3(-2)^2 - (-2) + 2 = 12 + 2 + 2 = 16.$

$f(a) = 3a^2 - a + 2.$

$f(-a) = 3(-a)^2 - (-a) + 2 = 3a^2 + a + 2.$

$f(a+1) = 3(a+1)^2 - (a+1) + 2 = 3(a^2 + 2a + 1) - a - 1 + 2 = 3a^2 + 6a + 3 - a + 1 = 3a^2 + 5a + 4.$

$2f(a) = 2 \cdot f(a) = 2(3a^2 - a + 2) = 6a^2 - 2a + 4.$

$f(2a) = 3(2a)^2 - (2a) + 2 = 3(4a^2) - 2a + 2 = 12a^2 - 2a + 2.$

$f(a^2) = 3(a^2)^2 - (a^2) + 2 = 3(a^4) - a^2 + 2 = 3a^4 - a^2 + 2.$

$$[f(a)]^2 = \left[3a^2 - a + 2\right]^2 = \left(3a^2 - a + 2\right)\left(3a^2 - a + 2\right)$$
$$= 9a^4 - 3a^3 + 6a^2 - 3a^3 + a^2 - 2a + 6a^2 - 2a + 4 = 9a^4 - 6a^3 + 13a^2 - 4a + 4.$$

$f(a+h) = 3(a+h)^2 - (a+h) + 2 = 3(a^2 + 2ah + h^2) - a - h + 2 = 3a^2 + 6ah + 3h^2 - a - h + 2.$

20. A spherical balloon with radius $r+1$ has volume $V(r+1) = \frac{4}{3}\pi(r+1)^3 = \frac{4}{3}\pi\left(r^3 + 3r^2 + 3r + 1\right)$. We wish to find the amount of air needed to inflate the balloon from a radius of r to $r+1$. Hence, we need to find the difference $V(r+1) - V(r) = \frac{4}{3}\pi\left(r^3 + 3r^2 + 3r + 1\right) - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(3r^2 + 3r + 1\right)$.

21. $f(x) = x - x^2$, so

$f(2+h) = 2 + h - (2+h)^2 = 2 + h - (4 + 4h + h^2) = 2 + h - 4 - 4h - h^2 = -\left(h^2 + 3h + 2\right),$

$f(x+h) = x + h - (x+h)^2 = x + h - (x^2 + 2xh + h^2) = x + h - x^2 - 2xh - h^2$, and

$$\frac{f(x+h) - f(x)}{h} = \frac{x + h - x^2 - 2xh - h^2 - x + x^2}{h} = \frac{h - 2xh - h^2}{h} = \frac{h(1 - 2x - h)}{h} = 1 - 2x - h.$$

22. $f(x) = \dfrac{x}{x+1}$, so $f(2+h) = \dfrac{2+h}{2+h+1} = \dfrac{2+h}{3+h}$, $f(x+h) = \dfrac{x+h}{x+h+1}$, and

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{x+h}{x+h+1} - \frac{x}{x+1}}{h} = \frac{(x+h)(x+1) - x(x+h+1)}{h(x+h+1)(x+1)} = \frac{1}{(x+h+1)(x+1)}.$$

23. $f(x) = x/(3x-1)$ is defined for all x except when $0 = 3x - 1 \quad \Leftrightarrow \quad x = \frac{1}{3}$, so the domain is $\left\{x \in \mathbb{R} \mid x \neq \frac{1}{3}\right\} = \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{3}, \infty\right)$.

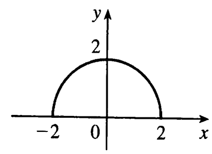
24. $f(x) = (5x+4)\,/\left(x^2+3x+2\right)$ is defined for all x except when $0 = x^2 + 3x + 2 \quad \Leftrightarrow \quad 0 = (x+2)(x+1)$ $\Leftrightarrow \quad x = -2$ or -1, so the domain is $\{x \in \mathbb{R} \mid x \neq -2, -1\} = (-\infty, -2) \cup (-2, -1) \cup (-1, \infty)$.

25. $f(t) = \sqrt{t} + \sqrt[3]{t}$ is defined when $t \geq 0$. These values of t give real number results for $\sqrt{t}$, whereas any value of t gives a real number result for $\sqrt[3]{t}$. The domain is $[0, \infty)$.

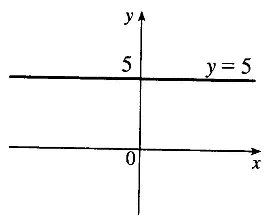
26. $g(u) = \sqrt{u} + \sqrt{4-u}$ is defined when $u \geq 0$ and $4 - u \geq 0 \quad \Leftrightarrow \quad u \leq 4$. Thus, the domain is $0 \leq u \leq 4 = [0, 4]$.

27. $h(x) = 1\,/\sqrt[4]{x^2 - 5x}$ is defined when $x^2 - 5x > 0 \quad \Leftrightarrow \quad x(x-5) > 0$. Note that $x^2 - 5x \neq 0$ since that would result in division by zero. The expression $x(x-5)$ is positive if $x < 0$ or $x > 5$. (See Appendix A for methods for solving inequalities.) Thus, the domain is $(-\infty, 0) \cup (5, \infty)$.

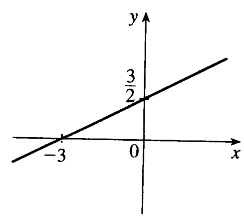
28. $h(x) = \sqrt{4 - x^2}$. Now $y = \sqrt{4-x^2} \quad \Rightarrow \quad y^2 = 4 - x^2 \quad \Leftrightarrow \quad x^2 + y^2 = 4$, so the graph is the top half of a circle of radius 2 with center at the origin. The domain is $\left\{x \mid 4 - x^2 \geq 0\right\} = \left\{x \mid 4 \geq x^2\right\} = \{x \mid 2 \geq |x|\} = [-2, 2]$. From the graph, the range is $0 \leq y \leq 2$, or $[0, 2]$.



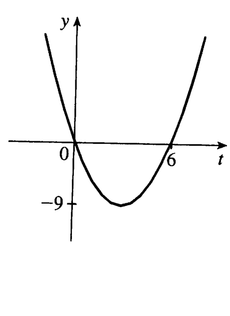
29. $f(x) = 5$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a horizontal line with y-intercept 5.



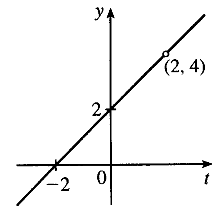
30. $F(x) = \frac{1}{2}(x+3)$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of F is a line with x-intercept -3 and y-intercept $\frac{3}{2}$.



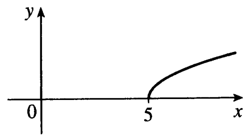
31. $f(t) = t^2 - 6t$ is defined for all real numbers, so the domain is $\mathbb{R}$, or $(-\infty, \infty)$. The graph of f is a parabola opening upward since the coefficient of t^2 is positive. To find the t-intercepts, let $y = 0$ and solve for t. $0 = t^2 - 6t = t(t-6) \quad \Rightarrow \quad t = 0$ and $t = 6$. The t-coordinate of the vertex is halfway between the t-intercepts, that is, at $t = 3$. Since $f(3) = 3^2 - 6 \cdot 3 = -9$, the vertex is $(3, -9)$.



32. $H(t) = \dfrac{4 - t^2}{2 - t} = \dfrac{(2+t)(2-t)}{2-t}$, so for $t \neq 2$, $H(t) = 2 + t$. The domain is $\{t \mid t \neq 2\}$. So the graph of H is the same as the graph of the function $f(t) = t + 2$ (a line) except for the hole at $(2, 4)$.

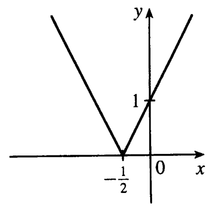


33. $g(x) = \sqrt{x - 5}$ is defined when $x - 5 \geq 0$ or $x \geq 5$, so the domain is $[5, \infty)$. Since $y = \sqrt{x-5} \;\Rightarrow\; y^2 = x - 5 \;\Rightarrow\; x = y^2 + 5$, we see that g is the top half of a parabola.



34. $$F(x) = |2x + 1| = \begin{cases} 2x + 1 & \text{if } 2x + 1 \geq 0 \\ -(2x+1) & \text{if } 2x + 1 < 1 \end{cases}$$
$$= \begin{cases} 2x + 1 & \text{if } x \geq -\frac{1}{2} \\ -2x - 1 & \text{if } x < -\frac{1}{2} \end{cases}$$

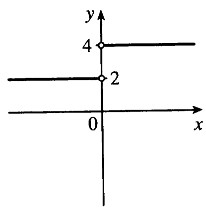
The domain is $\mathbb{R}$, or $(-\infty, \infty)$.



35. $G(x) = \dfrac{3x + |x|}{x}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

$$G(x) = \begin{cases} \dfrac{3x + x}{x} & \text{if } x > 0 \\[2ex] \dfrac{3x - x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} \dfrac{4x}{x} & \text{if } x > 0 \\[2ex] \dfrac{2x}{x} & \text{if } x < 0 \end{cases} = \begin{cases} 4 & \text{if } x > 0 \\ 2 & \text{if } x < 0 \end{cases}$$

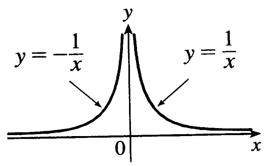
Note that G is not defined for $x = 0$. The domain is $(-\infty, 0) \cup (0, \infty)$.



36. $g(x) = \dfrac{|x|}{x^2}$. Since $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, we have

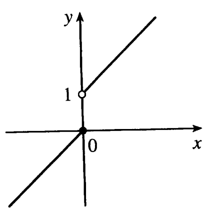
$$g(x) = \begin{cases} \dfrac{x}{x^2} & \text{if } x > 0 \\[2ex] \dfrac{-x}{x^2} & \text{if } x < 0 \end{cases} = \begin{cases} \dfrac{1}{x} & \text{if } x > 0 \\[2ex] \dfrac{1}{-x} & \text{if } x < 0 \end{cases}$$

Note that g is not defined for $x = 0$. The domain is $(-\infty, 0) \cup (0, \infty)$.



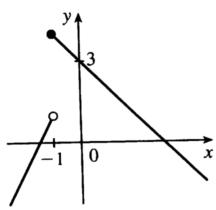
37. $f(x) = \begin{cases} x & \text{if } x \leq 0 \\ x + 1 & \text{if } x > 0 \end{cases}$

Domain is $\mathbb{R}$, or $(-\infty, \infty)$.



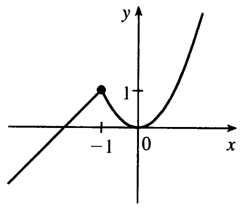
38. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \geq -1 \end{cases}$

Domain is $\mathbb{R}$, or $(-\infty, \infty)$.



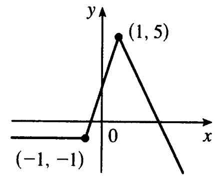
39. $f(x) = \begin{cases} x+2 & \text{if } x \le -1 \\ x^2 & \text{if } x > -1 \end{cases}$

Note that for $x = -1$, both $x + 2$ and x^2 are equal to 1. Domain is $\mathbb{R}$.



40. $f(x) = \begin{cases} -1 & \text{if } x \le -1 \\ 3x+2 & \text{if } -1 < x < 1 \\ 7-2x & \text{if } x \ge 1 \end{cases}$

Domain is $\mathbb{R}$.



41. Recall that the slope m of a line between the two points (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$ and an equation of the line connecting those two points is $y - y_1 = m(x - x_1)$. The slope of this line segment is $\dfrac{-6-1}{4-(-2)} = -\dfrac{7}{6}$, so an equation is $y - 1 = -\frac{7}{6}(x+2)$. The function is $f(x) = -\frac{7}{6}x - \frac{4}{3}$, $-2 \le x \le 4$.

42. The slope of this line segment is $\dfrac{3-(-2)}{6-(-3)} = \dfrac{5}{9}$, so an equation is $y + 2 = \frac{5}{9}(x+3)$. The function is $f(x) = \frac{5}{9}x - \frac{1}{3}$, $-3 \le x \le 6$.

43. We need to solve the given equation for y. $x + (y-1)^2 = 0 \;\Leftrightarrow\; (y-1)^2 = -x \;\Leftrightarrow\; y - 1 = \pm\sqrt{-x} \;\Leftrightarrow\; y = 1 \pm \sqrt{-x}$. The expression with the positive radical represents the top half of the parabola, and the one with the negative radical represents the bottom half. Hence, we want $f(x) = 1 - \sqrt{-x}$. Note that the domain is $x \le 0$.

44. $(x-1)^2 + y^2 = 1 \;\Leftrightarrow\; y = \pm\sqrt{1-(x-1)^2} = \pm\sqrt{2x - x^2}$. The top half is given by the function $f(x) = \sqrt{2x - x^2}$, $0 \le x \le 2$.

45. For $-1 \le x \le 2$, the graph is the line with slope 1 and y-intercept 1, that is, the line $y = x + 1$. For $2 < x \le 4$, the graph is the line with slope $-\frac{3}{2}$ and x-intercept 4 [which corresponds to the point $(4, 0)$], so $y - 0 = -\frac{3}{2}(x-4) = -\frac{3}{2}x + 6$. So the function is $f(x) = \begin{cases} x+1 & \text{if } -1 \le x \le 2 \\ -\frac{3}{2}x + 6 & \text{if } 2 < x \le 4 \end{cases}$

46. For $x \le 0$, the graph is the line $y = 2$. For $0 < x \le 1$, the graph is the line with slope -2 and y-intercept 2, that is, the line $y = -2x + 2$. For $x > 1$, the graph is the line with slope 1 and x-intercept 1, that is, the line $y = 1\,(x-1) = x - 1$. So the function is $f(x) = \begin{cases} 2 & \text{if } x \le 0 \\ -2x+2 & \text{if } 0 < x \le 1 \\ x-1 & \text{if } 1 < x \end{cases}$.

47. Let the length and width of the rectangle be L and W. Then the perimeter is $2L + 2W = 20$ and the area is $A = LW$. Solving the first equation for W in terms of L gives $W = \dfrac{20 - 2L}{2} = 10 - L$. Thus, $A(L) = L(10 - L) = 10L - L^2$. Since lengths are positive, the domain of A is $0 < L < 10$. If we further restrict L to be larger than W, then $5 < L < 10$ would be the domain.

48. Let the length and width of the rectangle be L and W. Then the area is $LW = 16$, so that $W = 16/L$. The perimeter is $P = 2L + 2W$, so $P(L) = 2L + 2(16/L) = 2L + 32/L$, and the domain of P is $L > 0$, since lengths must be positive quantities. If we further restrict L to be larger than W, then $L > 4$ would be the domain.

49. Let the length of a side of the equilateral triangle be x. Then by the Pythagorean Theorem, the height y of the triangle satisfies $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$, so that $y^2 = x^2 - \frac{1}{4}x^2 = \frac{3}{4}x^2$ and $y = \frac{\sqrt{3}}{2}x$. Using the formula for the area A of a triangle, $A = \frac{1}{2}(\text{base})(\text{height})$, we obtain $A(x) = \frac{1}{2}(x)\left(\frac{\sqrt{3}}{2}x\right) = \frac{\sqrt{3}}{4}x^2$, with domain $x > 0$.

50. Let the volume of the cube be V and the length of an edge be L. Then $V = L^3$ so $L = \sqrt[3]{V}$, and the surface area is $S(V) = 6\left(\sqrt[3]{V}\right)^2 = 6V^{2/3}$, with domain $V > 0$.

51. Let each side of the base of the box have length x, and let the height of the box be h. Since the volume is 2, we know that $2 = hx^2$, so that $h = 2/x^2$, and the surface area is $S = x^2 + 4xh$. Thus, $S(x) = x^2 + 4x(2/x^2) = x^2 + (8/x)$, with domain $x > 0$.

52. The area of the window is $A = xh + \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = xh + \dfrac{\pi x^2}{8}$, where h is the height of the rectangular portion of the window. The perimeter is $P = 2h + x + \frac{1}{2}\pi x = 30 \quad\Leftrightarrow\quad 2h = 30 - x - \frac{1}{2}\pi x \quad\Leftrightarrow\quad h = \frac{1}{4}(60 - 2x - \pi x)$. Thus,

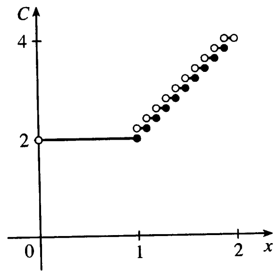
$$A(x) = x\,\frac{60 - 2x - \pi x}{4} + \frac{\pi x^2}{8} = 15x - \tfrac{1}{2}x^2 - \tfrac{\pi}{4}x^2 + \tfrac{\pi}{8}x^2 = 15x - \tfrac{4}{8}x^2 - \tfrac{\pi}{8}x^2 = 15x - x^2\left(\tfrac{\pi+4}{8}\right)$$

Since the lengths x and h must be positive quantities, we have $x > 0$ and $h > 0$. For $h > 0$, we have $2h > 0 \quad\Leftrightarrow\quad 30 - x - \frac{1}{2}\pi x > 0 \quad\Leftrightarrow\quad 60 > 2x + \pi x \quad\Leftrightarrow\quad x < \dfrac{60}{2+\pi}$. Hence, the domain of A is $0 < x < \dfrac{60}{2+\pi}$.

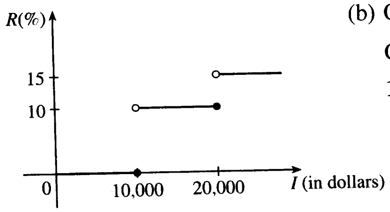
53. The height of the box is x and the length and width are $L = 20 - 2x$, $W = 12 - 2x$. Then $V = LWx$ and so $V(x) = (20 - 2x)(12 - 2x)(x) = 4(10 - x)(6 - x)(x) = 4x(60 - 16x + x^2) = 4x^3 - 64x^2 + 240x$. The sides L, W, and x must be positive. Thus, $L > 0 \quad\Leftrightarrow\quad 20 - 2x > 0 \quad\Leftrightarrow\quad x < 10$; $W > 0 \quad\Leftrightarrow\quad 12 - 2x > 0 \quad\Leftrightarrow\quad x < 6$; and $x > 0$. Combining these restrictions gives us the domain $0 < x < 6$.

54.

$$C(x) = \begin{cases} \$2.00 & \text{if } 0.0 < x \le 1.0 \\ 2.20 & \text{if } 1.0 < x \le 1.1 \\ 2.40 & \text{if } 1.1 < x \le 1.2 \\ 2.60 & \text{if } 1.2 < x \le 1.3 \\ 2.80 & \text{if } 1.3 < x \le 1.4 \\ 3.00 & \text{if } 1.4 < x \le 1.5 \\ 3.20 & \text{if } 1.5 < x \le 1.6 \\ 3.40 & \text{if } 1.6 < x \le 1.7 \\ 3.60 & \text{if } 1.7 < x \le 1.8 \\ 3.80 & \text{if } 1.8 < x \le 1.9 \\ 4.00 & \text{if } 1.9 < x < 2.0 \end{cases}$$

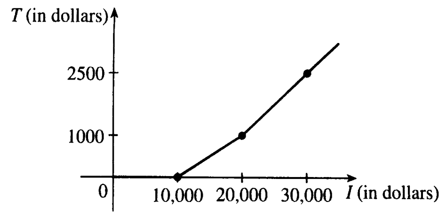


55. (a)



(b) On \$14,000, tax is assessed on \$4000, and 10%(\$4000) = \$400.
On \$26,000, tax is assessed on \$16,000, and
10%(\$10,000) + 15%(\$6000) = \$1000 + \$900 = \$1900.

(c) As in part (b), there is \$1000 tax assessed on \$20,000 of income, so the graph of T is a line segment from $(10{,}000, 0)$ to $(20{,}000, 1000)$. The tax on \$30,000 is \$2500, so the graph of T for $x > 20{,}000$ is the ray with initial point $(20{,}000, 1000)$ that passes through $(30{,}000, 2500)$.



56. One example is the amount paid for cable or telephone system repair in the home, usually measured to the nearest quarter hour. Another example is the amount paid by a student in tuition fees, if the fees vary according to the number of credits for which the student has registered.

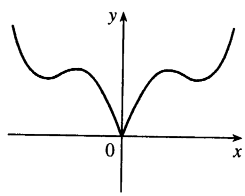
57. f is an odd function because its graph is symmetric about the origin. g is an even function because its graph is symmetric with respect to the y-axis.

58. f is not an even function since it is not symmetric with respect to the y-axis. f is not an odd function since it is not symmetric about the origin. Hence, f is *neither* even nor odd. g is an even function because its graph is symmetric with respect to the y-axis.

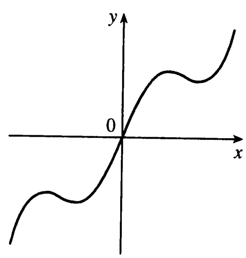
59. (a) Because an even function is symmetric with respect to the y-axis, and the point $(5, 3)$ is on the graph of this even function, the point $(-5, 3)$ must also be on its graph.

(b) Because an odd function is symmetric with respect to the origin, and the point $(5, 3)$ is on the graph of this odd function, the point $(-5, -3)$ must also be on its graph.

60. (a) If f is even, we get the rest of the graph by reflecting about the y-axis.



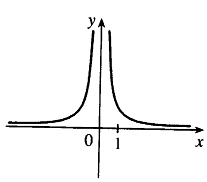
(b) If f is odd, we get the rest of the graph by rotating 180° about the origin.



61. $f(x) = x^{-2}$.

$$f(-x) = (-x)^{-2} = \frac{1}{(-x)^2} = \frac{1}{x^2}$$
$$= x^{-2} = f(x)$$

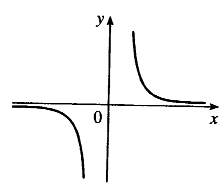
So f is an even function.



62. $f(x) = x^{-3}$.

$$f(-x) = (-x)^{-3} = \frac{1}{(-x)^3} = \frac{1}{-x^3}$$
$$= -\frac{1}{x^3} = -\left(x^{-3}\right) = -f(x)$$

So f is odd.



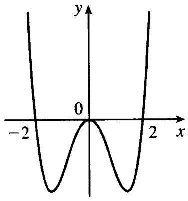
63. $f(x) = x^2 + x$, so $f(-x) = (-x)^2 + (-x) = x^2 - x$. Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.

64. $f(x) = x^4 - 4x^2$.

$f(-x) = (-x)^4 - 4(-x)^2$

$= x^4 - 4x^2 = f(x)$

So f is even.

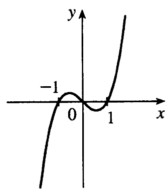


65. $f(x) = x^3 - x$.

$f(-x) = (-x)^3 - (-x) = -x^3 + x$

$= -\left(x^3 - x\right) = -f(x)$

So f is odd.



66. $f(x) = 3x^3 + 2x^2 + 1$, so $f(-x) = 3(-x)^3 + 2(-x)^2 + 1 = -3x^3 + 2x^2 + 1$. Since this is neither $f(x)$ nor $-f(x)$, the function f is neither even nor odd.

1.2 Mathematical Models: A Catalog of Essential Functions

1. (a) $f(x) = \sqrt[5]{x}$ is a root function with $n = 5$.

(b) $g(x) = \sqrt{1 - x^2}$ is an algebraic function because it is a root of a polynomial.

(c) $h(x) = x^9 + x^4$ is a polynomial of degree 9.

(d) $r(x) = \dfrac{x^2 + 1}{x^3 + x}$ is a rational function because it is a ratio of polynomials.

(e) $s(x) = \tan 2x$ is a trigonometric function.

(f) $t(x) = \log_{10} x$ is a logarithmic function.

2. (a) $y = (x - 6)/(x + 6)$ is a rational function because it is a ratio of polynomials.

(b) $y = x + x^2/\sqrt{x - 1}$ is an algebraic function because it involves polynomials and roots of polynomials.

(c) $y = 10^x$ is an exponential function (notice that x is the *exponent*).

(d) $y = x^{10}$ is a power function (notice that x is the *base*).

(e) $y = 2t^6 + t^4 - \pi$ is a polynomial of degree 6.

(f) $y = \cos\theta + \sin\theta$ is a trigonometric function.

3. We notice from the figure that g and h are even functions (symmetric with respect to the y-axis) and that f is an odd function (symmetric with respect to the origin). So (b) $\left[y = x^5\right]$ must be f. Since g is flatter than h near the origin, we must have (c) $\left[y = x^8\right]$ matched with g and (a) $\left[y = x^2\right]$ matched with h.

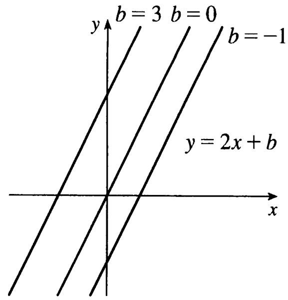
4. (a) The graph of $y = 3x$ is a line (choice G).

(b) $y = 3^x$ is an exponential function (choice f).

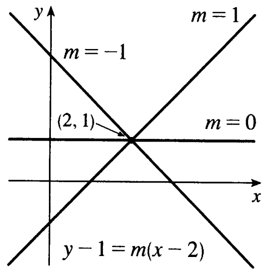
(c) $y = x^3$ is an odd polynomial function or power function (choice F).

(d) $y = \sqrt[3]{x} = x^{1/3}$ is a root function (choice g).

5. (a) An equation for the family of linear functions with slope 2 is $y = f(x) = 2x + b$, where b is the y-intercept.

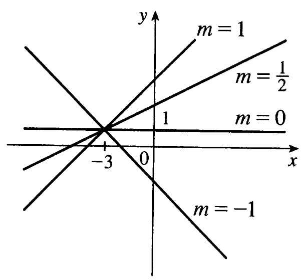


(b) $f(2) = 1$ means that the point $(2, 1)$ is on the graph of f. We can use the point-slope form of a line to obtain an equation for the family of linear functions through the point $(2, 1)$. $y - 1 = m(x - 2)$, which is equivalent to $y = mx + (1 - 2m)$ in slope-intercept form.

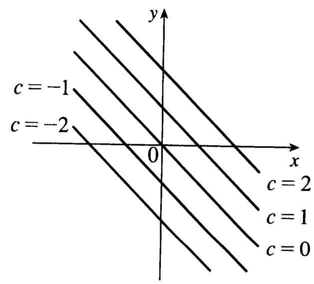


(c) To belong to both families, an equation must have slope $m = 2$, so the equation in part (b), $y = mx + (1 - 2m)$, becomes $y = 2x - 3$. It is the *only* function that belongs to both families.

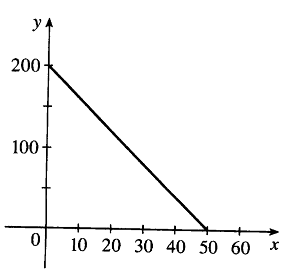
6. All members of the family of linear functions $f(x) = 1 + m(x + 3)$ have graphs that are lines passing through the point $(-3, 1)$.



7. All members of the family of linear functions $f(x) = c - x$ have graphs that are lines with slope -1. The y-intercept is c.

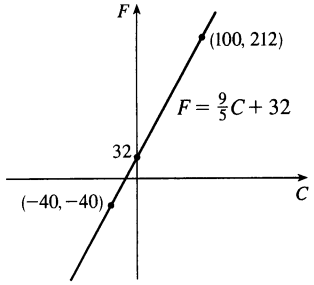


8. (a)



(b) The slope of -4 means that for each increase of 1 dollar for a rental space, the number of spaces rented *decreases* by 4. The y-intercept of 200 is the number of spaces that would be occupied if there were no charge for each space. The x-intercept of 50 is the smallest rental fee that results in no spaces rented.

9. (a)

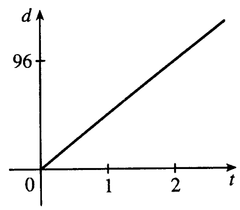


(b) The slope of $\frac{9}{5}$ means that F increases $\frac{9}{5}$ degrees for each increase of 1 °C. (Equivalently, F increases by 9 when C increases by 5 and F decreases by 9 when C decreases by 5.) The F-intercept of 32 is the Fahrenheit temperature corresponding to a Celsius temperature of 0.

10. (a) Let $d =$ distance traveled (in miles) and $t =$ time elapsed (in hours). At $t = 0$, $d = 0$ and at $t = 50$ minutes $= 50 \cdot \frac{1}{60} = \frac{5}{6}$ h, $d = 40$. Thus we have two points: $(0, 0)$ and $\left(\frac{5}{6}, 40\right)$, so

$$m = \frac{40 - 0}{\frac{5}{6} - 0} = 48 \text{ and so } d = 48t.$$

(b)



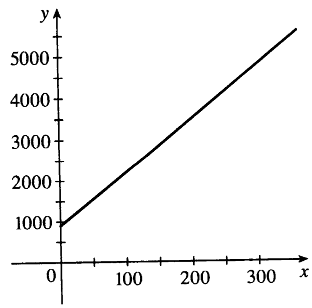
(c) The slope is 48 and represents the car's speed in mi/h.

11. (a) Using N in place of x and T in place of y, we find the slope to be $\dfrac{T_2 - T_1}{N_2 - N_1} = \dfrac{80 - 70}{173 - 113} = \dfrac{10}{60} = \dfrac{1}{6}$. So a linear equation is $T - 80 = \frac{1}{6}(N - 173) \iff T - 80 = \frac{1}{6}N - \frac{173}{6} \iff T = \frac{1}{6}N + \frac{307}{6}$ $\left[\frac{307}{6} = 51.1\overline{6}\right]$.

(b) The slope of $\frac{1}{6}$ means that the temperature in Fahrenheit degrees increases one-sixth as rapidly as the number of cricket chirps per minute. Said differently, each increase of 6 cricket chirps per minute corresponds to an increase of 1 °F.

(c) When $N = 150$, the temperature is given approximately by $T = \frac{1}{6}(150) + \frac{307}{6} = 76.1\overline{6}$ °F ≈ 76 °F.

12. (a) Let x denote the number of chairs produced in one day and y the associated cost. Using the points $(100, 2200)$ and $(300, 4800)$ we get the slope $\frac{4800-2200}{300-100} = \frac{2600}{200} = 13$. So $y - 2200 = 13(x - 100) \iff$ $y = 13x + 900$.



(b) The slope of the line in part (a) is 13 and it represents the cost (in dollars) of producing each additional chair.

(c) The y-intercept is 900 and it represents the fixed daily costs of operating the factory.

13. (a) We are given $\dfrac{\text{change in pressure}}{\text{10 feet change in depth}} = \dfrac{4.34}{10} = 0.434$. Using P for pressure and d for depth with the point $(d, P) = (0, 15)$, we have the slope-intercept form of the line, $P = 0.434d + 15$.

(b) When $P = 100$, then $100 = 0.434d + 15 \iff 0.434d = 85 \iff d = \frac{85}{0.434} \approx 195.85$ feet. Thus, the pressure is 100 lb/in^2 at a depth of approximately 196 feet.

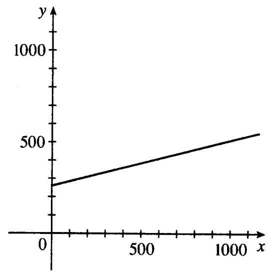
14. (a) Using d in place of x and C in place of y, we find the slope to be

$$\frac{C_2 - C_1}{d_2 - d_1} = \frac{460 - 380}{800 - 480} = \frac{80}{320} = \frac{1}{4}$$

So a linear equation is $C - 460 = \frac{1}{4}(d - 800) \Leftrightarrow$ $C - 460 = \frac{1}{4}d - 200 \Leftrightarrow C = \frac{1}{4}d + 260$.

(b) Letting $d = 1500$ we get $C = \frac{1}{4}(1500) + 260 = 635$. The cost of driving 1500 miles is \$635.

(c)



The slope of the line represents the cost per mile, \$0.25.

(d) The y-intercept represents the fixed cost, \$260.

(e) A linear function gives a suitable model in this situation because you have fixed monthly costs such as insurance and car payments, as well as costs that increase as you drive, such as gasoline, oil, and tires, and the cost of these for each additional mile driven is a constant.

15. (a) The data appear to be periodic and a sine or cosine function would make the best model. A model of the form $f(x) = a\cos(bx) + c$ seems appropriate.

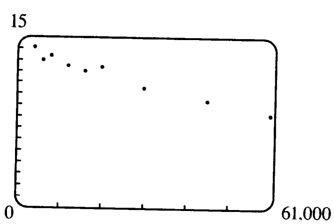
(b) The data appear to be decreasing in a linear fashion. A model of the form $f(x) = mx + b$ seems appropriate.

16. (a) The data appear to be increasing exponentially. A model of the form $f(x) = a \cdot b^x$ or $f(x) = a \cdot b^x + c$ seems appropriate.

(b) The data appear to be decreasing similarly to the values of the reciprocal function. A model of the form $f(x) = a/x$ seems appropriate.

Some values are given to many decimal places. These are the results given by several computer algebra systems—rounding is left to the reader.

17. (a)

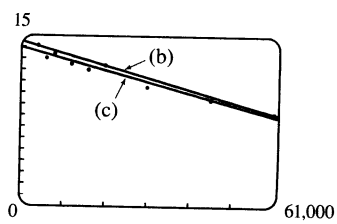


A linear model does seem appropriate.

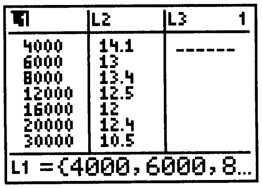
(b) Using the points $(4000, 14.1)$ and $(60{,}000, 8.2)$, we obtain

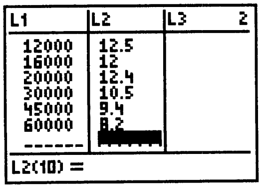
$$y - 14.1 = \frac{8.2 - 14.1}{60{,}000 - 4000}(x - 4000) \text{ or, equivalently,}$$

$y \approx -0.000105357x + 14.521429$.

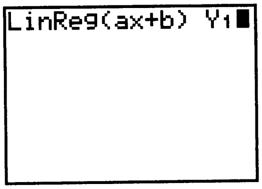


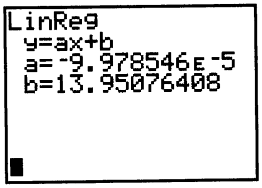
(c) Using a computing device, we obtain the least squares regression line $y = -0.0000997855x + 13.950764$. The following commands and screens illustrate how to find the least squares regression line on a TI-83 Plus. Enter the data into list one (L1) and list two (L2). Press [STAT] [1] to enter the editor.

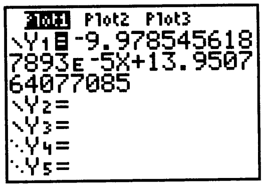




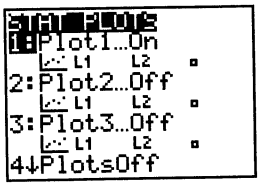
Find the regession line and store it in Y_1. Press [2nd] [QUIT] [STAT] [▶] [4] [VARS] [▶] [1] [1] [ENTER].

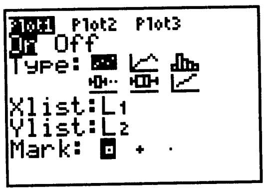




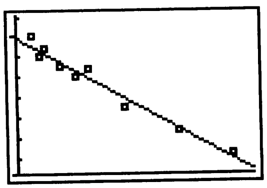


Note from the last figure that the regression line has been stored in Y_1 and that Plot1 has been turned on (Plot1 is highlighted). You can turn on Plot1 from the Y= menu by placing the cursor on Plot1 and pressing [ENTER] or by pressing [2nd] [STAT PLOT] [1] [ENTER].





Now press [ZOOM] [9] to produce a graph of the data and the regression line. Note that choice 9 of the ZOOM menu automatically selects a window that displays all of the data.

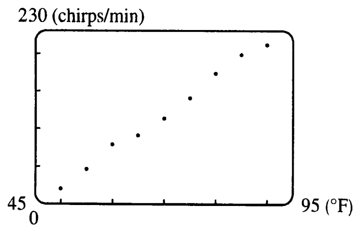


(d) When $x = 25{,}000$, $y \approx 11.456$; or about 11.5 per 100 population.

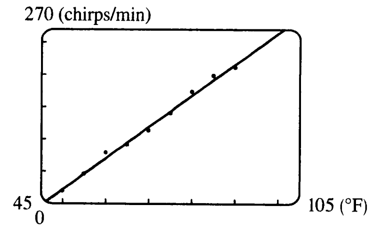
(e) When $x = 80{,}000$, $y \approx 5.968$; or about a 6% chance.

(f) When $x = 200{,}000$, y is negative, so the model does not apply.

18. (a)



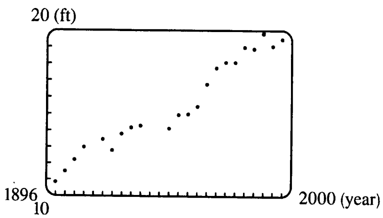
(b)



Using a computing device, we obtain the least squares regression line $y = 4.85\overline{6}x - 220.9\overline{6}$.

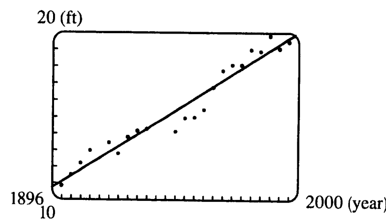
(c) When $x = 100°$ F, $y = 264.7 \approx 265$ chirps/min.

19. (a)



A linear model does seem appropriate.

(b)

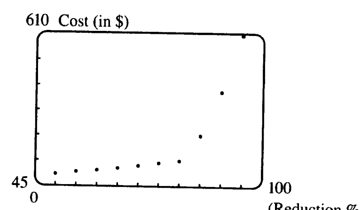


Using a computing device, we obtain the least squares regression line $y = 0.089119747x - 158.2403249$, where x is the year and y is the height in feet.

(c) When $x = 2000$, the model gives $y \approx 20.00$ ft. Note that the actual winning height for the 2000 Olympics is *less than* the winning height for 1996—so much for that prediction.

(d) When $x = 2100$, $y \approx 28.91$ ft. This would be an increase of 9.49 ft from 1996 to 2100. Even though there was an increase of 8.59 ft from 1900 to 1996, it is unlikely that a similar increase will occur over the next 100 years.

20. By looking at the scatter plot of the data, we rule out the linear and logarithmic models.



Scatter plot

We try various models:

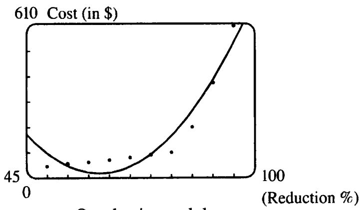
Quadratic: $y = 0.49\overline{6}x^2 - 62.2\overline{893}x + 1970.6\overline{39}$

Cubic: $y = 0.0201243201x^3 - 3.88037296x^2 + 247.6754468x - 5163.935198$

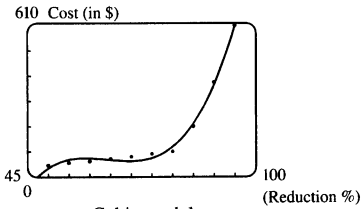
Quartic: $y = 0.0002951049x^4 - 0.0654560995x^3 + 5.27525641x^2 - 180.2266511x + 2203.210956$

Exponential: $y = 2.41422994\,(1.054516914)^x$

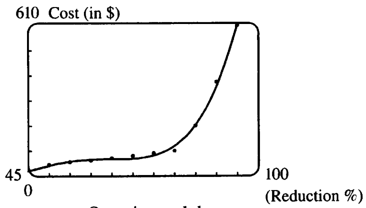
Power: $y = 0.000022854971x^{3.616078251}$



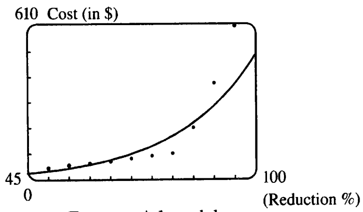
Quadratic model



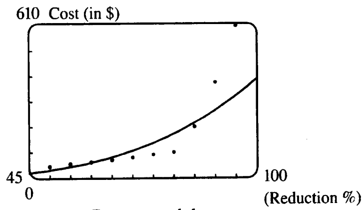
Cubic model



Quartic model



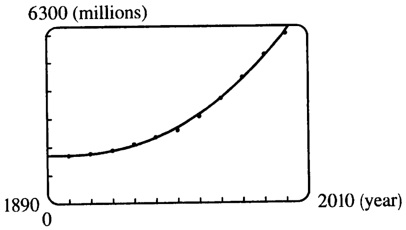
Exponential model



Power model

After examining the graphs of these models, we see that the cubic and quartic models are clearly the best.

21.



Using a computing device, we obtain the cubic function $y = ax^3 + bx^2 + cx + d$ with $a = 0.0012937$, $b = -7.06142$, $c = 12{,}823$, and $d = -7{,}743{,}770$. When $x = 1925$, $y \approx 1914$ (million).

22. (a) $T = 1.000396048d^{1.499661718}$

(b) The power model in part (a) is approximately $T = d^{1.5}$. Squaring both sides gives us $T^2 = d^3$, so the model matches Kepler's Third Law, $T^2 = kd^3$.

1.3 New Functions from Old Functions

1. (a) If the graph of f is shifted 3 units upward, its equation becomes $y = f(x) + 3$.

(b) If the graph of f is shifted 3 units downward, its equation becomes $y = f(x) - 3$.

(c) If the graph of f is shifted 3 units to the right, its equation becomes $y = f(x - 3)$.

(d) If the graph of f is shifted 3 units to the left, its equation becomes $y = f(x + 3)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 3, its equation becomes $y = 3f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 3, its equation becomes $y = \frac{1}{3}f(x)$.

2. (a) To obtain the graph of $y = 5f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5.

(b) To obtain the graph of $y = f(x - 5)$ from the graph of $y = f(x)$, shift the graph 5 units to the right.

(c) To obtain the graph of $y = -f(x)$ from the graph of $y = f(x)$, reflect the graph about the x-axis.

(d) To obtain the graph of $y = -5f(x)$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5 and reflect it about the x-axis.

(e) To obtain the graph of $y = f(5x)$ from the graph of $y = f(x)$, shrink the graph horizontally by a factor of 5.

(f) To obtain the graph of $y = 5f(x) - 3$ from the graph of $y = f(x)$, stretch the graph vertically by a factor of 5 and shift it 3 units downward.

3. (a) (graph 3) The graph of f is shifted 4 units to the right and has equation $y = f(x - 4)$.

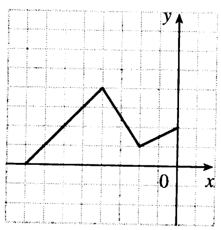
(b) (graph 1) The graph of f is shifted 3 units upward and has equation $y = f(x) + 3$.

(c) (graph 4) The graph of f is shrunk vertically by a factor of 3 and has equation $y = \frac{1}{3}f(x)$.

(d) (graph 5) The graph of f is shifted 4 units to the left and reflected about the x-axis. Its equation is $y = -f(x + 4)$.

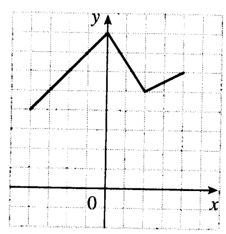
(e) (graph 2) The graph of f is shifted 6 units to the left and stretched vertically by a factor of 2. Its equation is $y = 2f(x + 6)$.

4. (a) To graph $y = f(x + 4)$ we shift the graph of f, 4 units to the left.



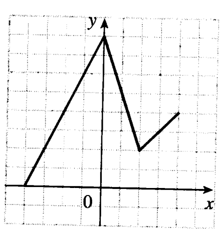
The point $(2, 1)$ on the graph of f corresponds to the point $(2 - 4, 1) = (-2, 1)$.

(b) To graph $y = f(x) + 4$ we shift the graph of f, 4 units upward.



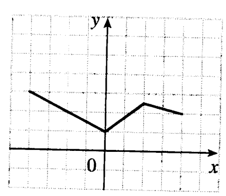
The point $(2, 1)$ on the graph of f corresponds to the point $(2, 1 + 4) = (2, 5)$.

(c) To graph $y = 2f(x)$ we stretch the graph of f vertically by a factor of 2.



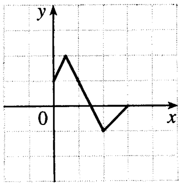
The point $(2, 1)$ on the graph of f corresponds to the point $(2, 2 \cdot 1) = (2, 2)$.

(d) To graph $y = -\frac{1}{2}f(x) + 3$, we shrink the graph of f vertically by a factor of 2, then reflect the resulting graph about the x-axis, then shift the resulting graph 3 units upward.



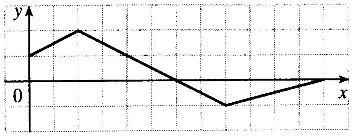
The point $(2, 1)$ on the graph of f corresponds to the point $\left(2, -\frac{1}{2} \cdot 1 + 3\right) = (2, 2.5)$.

5. (a) To graph $y = f(2x)$ we shrink the graph of f horizontally by a factor of 2.



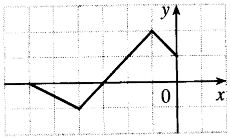
The point $(4, -1)$ on the graph of f corresponds to the point $\left(\frac{1}{2} \cdot 4, -1\right) = (2, -1)$.

(b) To graph $y = f\left(\frac{1}{2}x\right)$ we stretch the graph of f horizontally by a factor of 2.



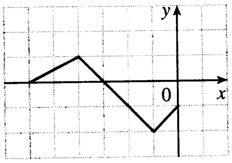
The point $(4, -1)$ on the graph of f corresponds to the point $(2 \cdot 4, -1) = (8, -1)$.

(c) To graph $y = f(-x)$ we reflect the graph of f about the y-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1) = (-4, -1)$.

(d) To graph $y = -f(-x)$ we reflect the graph of f about the y-axis, then about the x-axis.



The point $(4, -1)$ on the graph of f corresponds to the point $(-1 \cdot 4, -1 \cdot -1) = (-4, 1)$.

6. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 2 units to the right and stretched vertically by a factor of 2. Thus, a function describing the graph is

$$\begin{aligned} y &= 2f(x-2) = 2\sqrt{3(x-2) - (x-2)^2} \\ &= 2\sqrt{3x - 6 - (x^2 - 4x + 4)} = 2\sqrt{-x^2 + 7x - 10} \end{aligned}$$

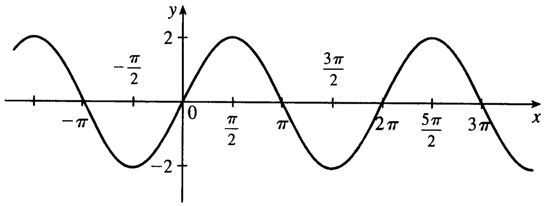
7. The graph of $y = f(x) = \sqrt{3x - x^2}$ has been shifted 4 units to the left, reflected about the x-axis, and shifted downward 1 unit. Thus, a function describing the graph is

$$y = \underbrace{-1 \cdot}_{\substack{\text{reflect} \\ \text{about} \\ x\text{-axis}}} \underbrace{f(x+4)}_{\substack{\text{shift} \\ \text{4 units} \\ \text{left}}} \underbrace{-\,1}_{\substack{\text{shift} \\ \text{1 unit} \\ \text{down}}}$$

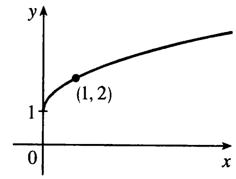
This function can be written as

$$\begin{aligned} y &= -f(x+4) - 1 = -\sqrt{3(x+4) - (x+4)^2} - 1 = -\sqrt{3x + 12 - (x^2 + 8x + 16)} - 1 \\ &= -\sqrt{-x^2 - 5x - 4} - 1 \end{aligned}$$

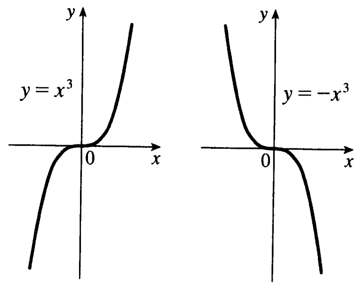
8. (a) The graph of $y = 2\sin x$ can be obtained from the graph of $y = \sin x$ by stretching it vertically by a factor of 2.



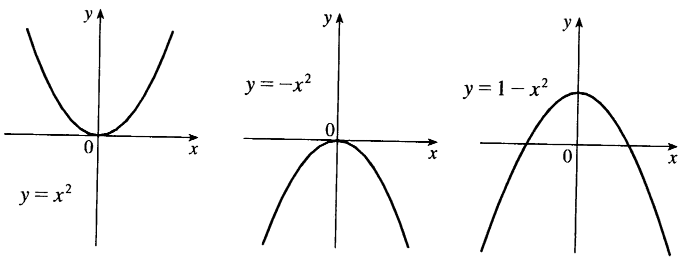
(b) The graph of $y = 1 + \sqrt{x}$ can be obtained from the graph of $y = \sqrt{x}$ by shifting it upward 1 unit.



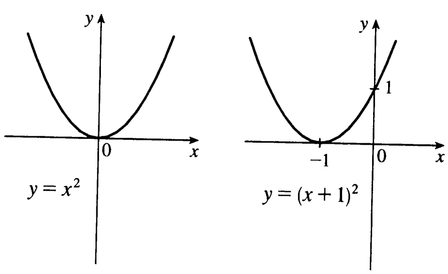
9. $y = -x^3$: Start with the graph of $y = x^3$ and reflect about the x-axis. Note: Reflecting about the y-axis gives the same result since substituting $-x$ for x gives us $y = (-x)^3 = -x^3$.



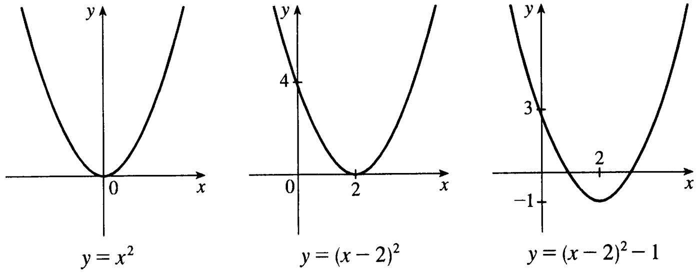
10. $y = 1 - x^2 = -x^2 + 1$: Start with the graph of $y = x^2$, reflect about the x-axis, and then shift 1 unit upward.



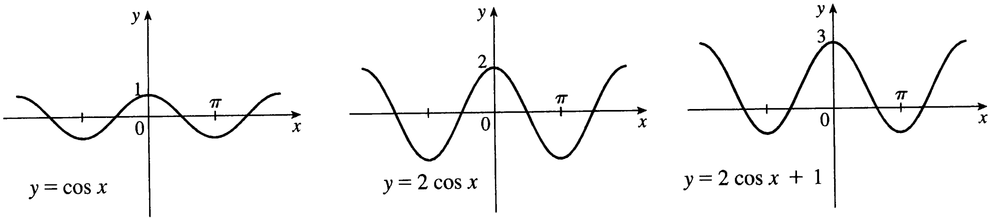
11. $y = (x + 1)^2$: Start with the graph of $y = x^2$ and shift 1 unit to the left.



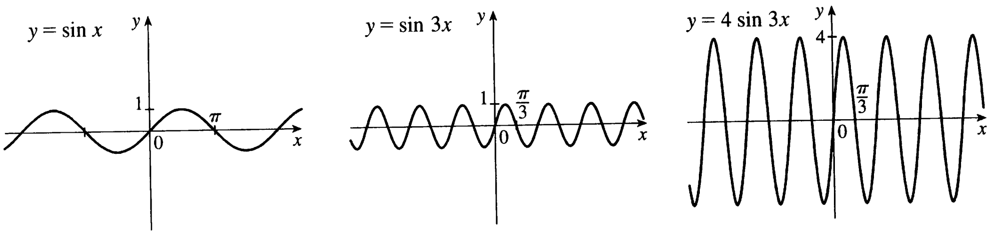
12. $y = x^2 - 4x + 3 = (x^2 - 4x + 4) - 1 = (x-2)^2 - 1$: Start with the graph of $y = x^2$, shift 2 units to the right, and then shift 1 unit downward.



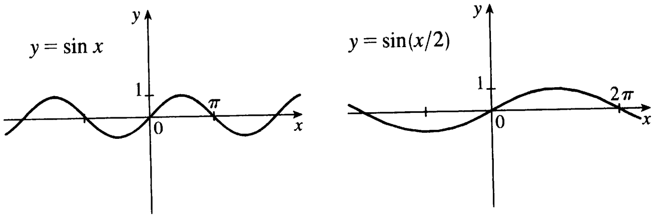
13. $y = 1 + 2\cos x$: Start with the graph of $y = \cos x$, stretch vertically by a factor of 2, and then shift 1 unit upward.



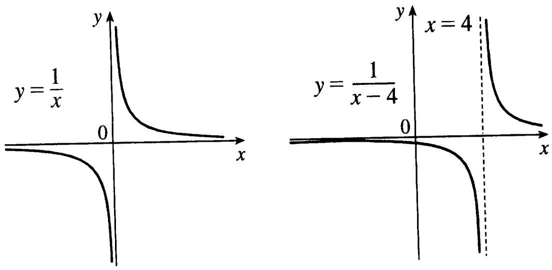
14. $y = 4\sin 3x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 3, and then stretch vertically by a factor of 4.



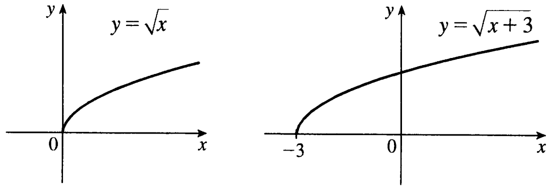
15. $y = \sin(x/2)$: Start with the graph of $y = \sin x$ and stretch horizontally by a factor of 2.



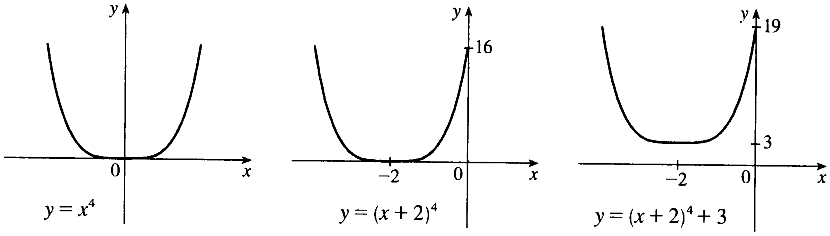
16. $y = 1/(x-4)$: Start with the graph of $y = 1/x$ and shift 4 units to the right.



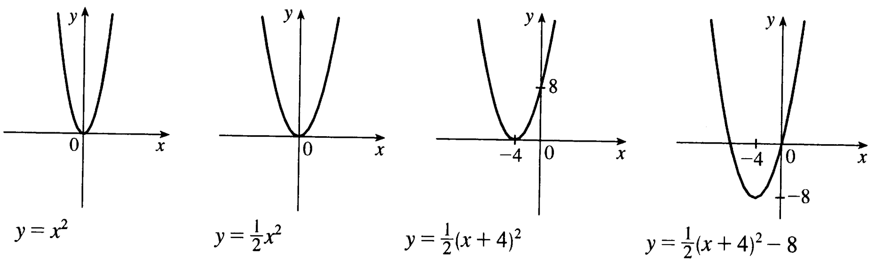
17. $y = \sqrt{x+3}$: Start with the graph of $y = \sqrt{x}$ and shift 3 units to the left.



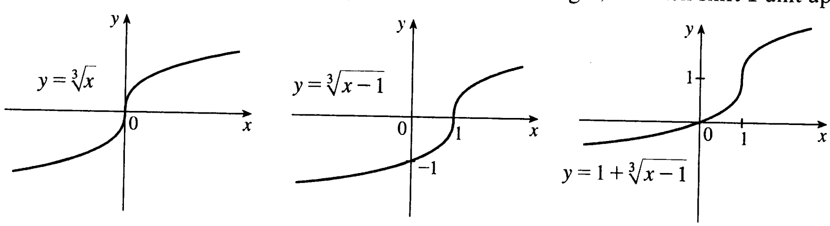
18. $y = (x+2)^4 + 3$: Start with the graph of $y = x^4$, shift 2 units to the left, and then shift 3 units upward.



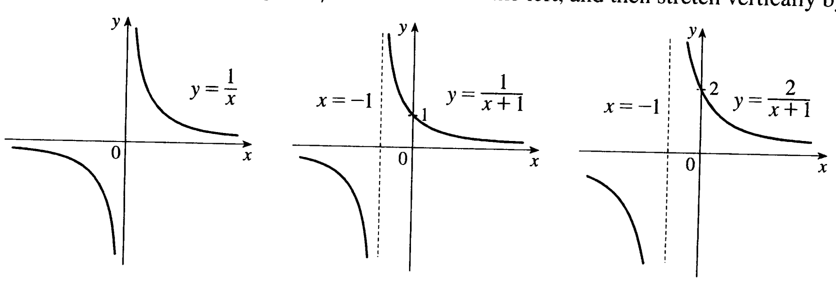
19. $y = \frac{1}{2}(x^2 + 8x) = \frac{1}{2}(x^2 + 8x + 16) - 8 = \frac{1}{2}(x+4)^2 - 8$: Start with the graph of $y = x^2$, compress vertically by a factor of 2, shift 4 units to the left, and then shift 8 units downward.



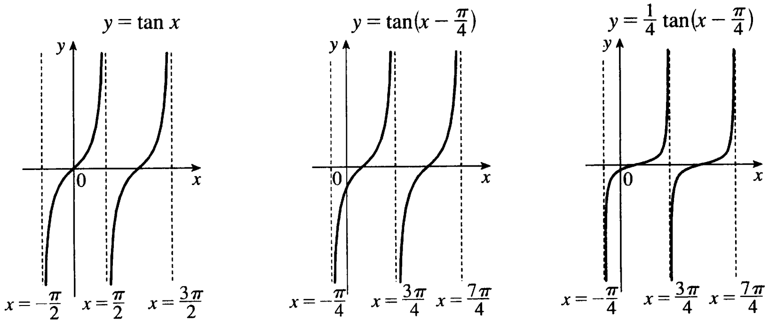
20. $y = 1 + \sqrt[3]{x-1}$: Start with the graph of $y = \sqrt[3]{x}$, shift 1 unit to the right, and then shift 1 unit upward.



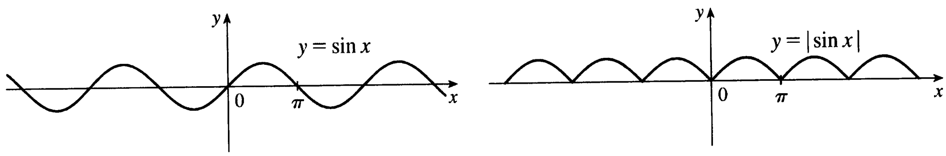
21. $y = 2/(x+1)$: Start with the graph of $y = 1/x$, shift 1 unit to the left, and then stretch vertically by a factor of 2.



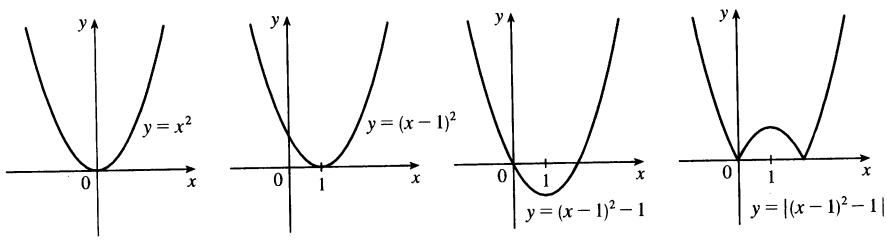
22. $y = \frac{1}{4}\tan(x - \frac{\pi}{4})$: Start with the graph of $y = \tan x$, shift $\frac{\pi}{4}$ units to the right, and then compress vertically by a factor of 4.



23. $y = |\sin x|$: Start with the graph of $y = \sin x$ and reflect all the parts of the graph below the x-axis about the x-axis.



24. $y = \left|x^2 - 2x\right| = \left|x^2 - 2x + 1 - 1\right| = \left|(x-1)^2 - 1\right|$: Start with the graph of $y = x^2$, shift 1 unit right, shift 1 unit downward, and reflect the portion of the graph below the x-axis about the x-axis.

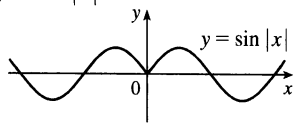


25. This is just like the solution to Example 4 except the amplitude of the curve (the 30°N curve in Figure 9 on June 21) is $14 - 12 = 2$. So the function is $L(t) = 12 + 2\sin\left[\frac{2\pi}{365}(t - 80)\right]$. March 31 is the 90th day of the year, so the model gives $L(90) \approx 12.34$ h. The daylight time (5:51 A.M. to 6:18 P.M.) is 12 hours and 27 minutes, or 12.45 h. The model value differs from the actual value by $\frac{12.45 - 12.34}{12.45} \approx 0.009$, less than 1%.

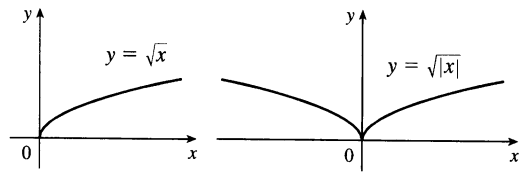
26. Using a sine function to model the brightness of Delta Cephei as a function of time, we take its period to be 5.4 days, its amplitude to be 0.35 (on the scale of magnitude), and its average magnitude to be 4.0. If we take $t = 0$ at a time of average brightness, then the magnitude (brightness) as a function of time t in days can be modeled by the formula $M(t) = 4.0 + 0.35\sin\left(\frac{2\pi}{5.4}t\right)$.

27. (a) To obtain $y = f(|x|)$, the portion of the graph of $y = f(x)$ to the right of the y-axis is reflected about the y-axis.

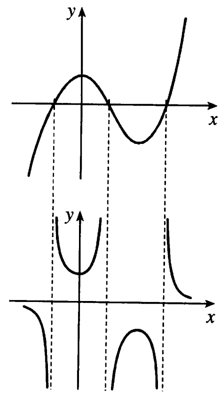
(b) $y = \sin|x|$



(c) $y = \sqrt{|x|}$

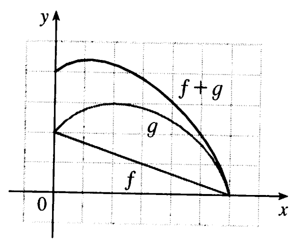


28. The most important features of the given graph are the x-intercepts and the maximum and minimum points. The graph of $y = 1/f(x)$ has vertical asymptotes at the x-values where there are x-intercepts on the graph of $y = f(x)$. The maximum of 1 on the graph of $y = f(x)$ corresponds to a minimum of $1/1 = 1$ on $y = 1/f(x)$. Similarly, the minimum on the graph of $y = f(x)$ corresponds to a maximum on the graph of $y = 1/f(x)$. As the values of y get large (positively or negatively) on the graph of $y = f(x)$, the values of y get close to zero on the graph of $y = 1/f(x)$.



29. Assuming that successive horizontal and vertical gridlines are a unit apart, we can make a table of approximate values as follows.

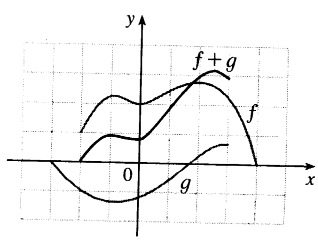
x	0	1	2	3	4	5	6
$f(x)$	2	1.7	1.3	1.0	0.7	0.3	0
$g(x)$	2	2.7	3	2.8	2.4	1.7	0
$f(x)+g(x)$	4	4.4	4.3	3.8	3.1	2.0	0



Connecting the points $(x, f(x) + g(x))$ with a smooth curve gives an approximation to the graph of $f + g$. Extra points can be plotted between those listed above if necessary.

30. First note that the domain of $f + g$ is the intersection of the domains of f and g; that is, $f + g$ is only defined where both f and g are defined. Taking the horizontal and vertical units of length to be the distances between successive vertical and horizontal gridlines, we can make a table of approximate values as follows:

x	-2	-1	0	1	2	2.5	3
$f(x)$	-1	2.2	2.0	2.4	2.7	2.7	2.3
$g(x)$	1	-1.3	-1.2	-0.6	0.3	0.5	0.7
$f(x)+g(x)$	0	0.9	0.8	1.8	3.0	3.2	3.0



Extra values of x (like the value 2.5 in the table above) can be added as needed.

31. $f(x) = x^3 + 2x^2$; $g(x) = 3x^2 - 1$. $D = \mathbb{R}$ for both f and g.

$(f+g)(x) = (x^3 + 2x^2) + (3x^2 - 1) = x^3 + 5x^2 - 1, \quad D = \mathbb{R}.$

$(f-g)(x) = (x^3 + 2x^2) - (3x^2 - 1) = x^3 - x^2 + 1, \quad D = \mathbb{R}.$

$(fg)(x) = (x^3 + 2x^2)(3x^2 - 1) = 3x^5 + 6x^4 - x^3 - 2x^2, \quad D = \mathbb{R}.$

$\left(\dfrac{f}{g}\right)(x) = \dfrac{x^3 + 2x^2}{3x^2 - 1}, \quad D = \left\{x \mid x \neq \pm\dfrac{1}{\sqrt{3}}\right\}$ since $3x^2 - 1 \neq 0$.

32. $f(x) = \sqrt{1+x}, \quad D = [-1, \infty); \quad g(x) = \sqrt{1-x}, \quad D = (-\infty, 1].$

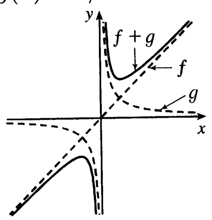
$(f+g)(x) = \sqrt{1+x} + \sqrt{1-x}, \quad D = (-\infty, 1] \cap [-1, \infty) = [-1, 1].$

$(f-g)(x) = \sqrt{1+x} - \sqrt{1-x}, \quad D = [-1, 1].$

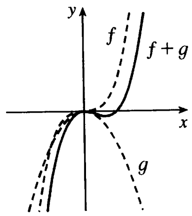
$(fg)(x) = \sqrt{1+x} \cdot \sqrt{1-x} = \sqrt{1-x^2}, \quad D = [-1, 1].$

$\left(\dfrac{f}{g}\right)(x) = \dfrac{\sqrt{1+x}}{\sqrt{1-x}}, \quad D = [-1, 1)$. We must exclude $x = 1$ since it would make $\dfrac{f}{g}$ undefined.

33. $f(x) = x, \quad g(x) = 1/x$



34. $f(x) = x^3, \quad g(x) = -x^2$



35. $f(x) = 2x^2 - x$; $g(x) = 3x + 2$. $D = \mathbb{R}$ for both f and g, and hence for their composites.

$(f \circ g)(x) = f(g(x)) = f(3x+2) = 2(3x+2)^2 - (3x+2) = 2(9x^2 + 12x + 4) - 3x - 2 = 18x^2 + 21x + 6.$

$(g \circ f)(x) = g(f(x)) = g(2x^2 - x) = 3(2x^2 - x) + 2 = 6x^2 - 3x + 2.$

$(f \circ f)(x) = f(f(x)) = f(2x^2 - x) = 2(2x^2 - x)^2 - (2x^2 - x) = 2(4x^4 - 4x^3 + x^2) - 2x^2 + x = 8x^4 - 8x^3 + x.$

$(g \circ g)(x) = g(g(x)) = g(3x+2) = 3(3x+2) + 2 = 9x + 6 + 2 = 9x + 8.$

36. $f(x) = 1 - x^3$, $D = \mathbb{R}$; $g(x) = 1/x$, $D = \{x \mid x \neq 0\}$.

$(f \circ g)(x) = f(g(x)) = f(1/x) = 1 - (1/x)^3 = 1 - 1/x^3$, $D = \{x \mid x \neq 0\}$.

$(g \circ f)(x) = g(f(x)) = g(1 - x^3) = 1/(1 - x^3)$, $D = \{x \mid 1 - x^3 \neq 0\} = \{x \mid x \neq 1\}$.

$(f \circ f)(x) = f(f(x)) = f(1 - x^3) = 1 - (1 - x^3)^3 \quad [= x^9 - 3x^6 + 3x^3]$, $D = \mathbb{R}$.

$(g \circ g)(x) = g(g(x)) = g(1/x) = 1/(1/x) = x$, $D = \{x \mid x \neq 0\}$ because 0 is not in the domain of g.

37. $f(x) = \sin x, \quad D = \mathbb{R}; \quad g(x) = 1 - \sqrt{x}, \quad D = [0, \infty).$

$(f \circ g)(x) = f(g(x)) = f(1 - \sqrt{x}\,) = \sin(1 - \sqrt{x}\,)$, $D = [0, \infty)$.

$(g \circ f)(x) = g(f(x)) = g(\sin x) = 1 - \sqrt{\sin x}$. For $\sqrt{\sin x}$ to be defined, we must have

$\sin x \geq 0 \quad \Leftrightarrow \quad x \in [0, \pi] \cup [2\pi, 3\pi] \cup [-2\pi, -\pi] \cup [4\pi, 5\pi] \cup [-4\pi, -3\pi] \cup \ldots$, so

$D = \{x \mid x \in [2n\pi, \pi + 2n\pi]$, where n is an integer$\}$.

$(f\circ f)(x)=f(f(x))=f(\sin x)=\sin(\sin x)$, $D=\mathbb{R}$.

$(g\circ g)(x)=g(g(x))=g(1-\sqrt{x})=1-\sqrt{1-\sqrt{x}}$,
$D=\{x\geq 0\mid 1-\sqrt{x}\geq 0\}=\{x\geq 0\mid 1\geq\sqrt{x}\}=\{x\geq 0\mid \sqrt{x}\leq 1\}=[0,1]$.

38. $f(x)=1-3x$, $D=\mathbb{R}$; $g(x)=5x^2+3x+2$, $D=\mathbb{R}$.

$$\begin{aligned}(f\circ g)(x)&=f(g(x))=f(5x^2+3x+2)=1-3(5x^2+3x+2)\\&=1-15x^2-9x-6=-15x^2-9x-5,\quad D=\mathbb{R}.\end{aligned}$$

$$\begin{aligned}(g\circ f)(x)&=g(f(x))=g(1-3x)=5(1-3x)^2+3(1-3x)+2=5(1-6x+9x^2)+3-9x+2\\&=5-30x+45x^2-9x+5=45x^2-39x+10,\quad D=\mathbb{R}.\end{aligned}$$

$$(f\circ f)(x)=f(f(x))=f(1-3x)=1-3(1-3x)=1-3+9x=9x-2,\quad D=\mathbb{R}.$$

$$\begin{aligned}(g\circ g)(x)&=g(g(x))=g(5x^2+3x+2)=5(5x^2+3x+2)^2+3(5x^2+3x+2)+2\\&=5(25x^4+30x^3+29x^2+12x+4)+15x^2+9x+6+2\\&=125x^4+150x^3+145x^2+60x+20+15x^2+9x+8\\&=125x^4+150x^3+160x^2+69x+28,\quad D=\mathbb{R}.\end{aligned}$$

39. $f(x)=x+\dfrac{1}{x}$, $D=\{x\mid x\neq 0\}$; $g(x)=\dfrac{x+1}{x+2}$, $D=\{x\mid x\neq -2\}$.

$$\begin{aligned}(f\circ g)(x)&=f(g(x))=f\left(\frac{x+1}{x+2}\right)=\frac{x+1}{x+2}+\frac{1}{\frac{x+1}{x+2}}=\frac{x+1}{x+2}+\frac{x+2}{x+1}\\&=\frac{(x+1)(x+1)+(x+2)(x+2)}{(x+2)(x+1)}=\frac{(x^2+2x+1)+(x^2+4x+4)}{(x+2)(x+1)}=\frac{2x^2+6x+5}{(x+2)(x+1)}\end{aligned}$$

Since $g(x)$ is not defined for $x=-2$ and $f(g(x))$ is not defined for $x=-2$ and $x=-1$, the domain of $(f\circ g)(x)$ is $D=\{x\mid x\neq -2,-1\}$.

$$(g\circ f)(x)=g(f(x))=g\left(x+\frac{1}{x}\right)=\frac{\left(x+\frac{1}{x}\right)+1}{\left(x+\frac{1}{x}\right)+2}=\frac{\dfrac{x^2+1+x}{x}}{\dfrac{x^2+1+2x}{x}}=\frac{x^2+x+1}{x^2+2x+1}=\frac{x^2+x+1}{(x+1)^2}.$$

Since $f(x)$ is not defined for $x=0$ and $g(f(x))$ is not defined for $x=-1$, the domain of $(g\circ f)(x)$ is $D=\{x\mid x\neq -1,0\}$.

$$\begin{aligned}(f\circ f)(x)&=f(f(x))=f\left(x+\frac{1}{x}\right)=\left(x+\frac{1}{x}\right)+\frac{1}{x+\frac{1}{x}}=x+\frac{1}{x}+\frac{1}{\frac{x^2+1}{x}}=x+\frac{1}{x}+\frac{x}{x^2+1}\\&=\frac{x(x)(x^2+1)+1\,(x^2+1)+x(x)}{x(x^2+1)}=\frac{x^4+x^2+x^2+1+x^2}{x(x^2+1)}\\&=\frac{x^4+3x^2+1}{x(x^2+1)},\quad D=\{x\mid x\neq 0\}.\end{aligned}$$

$$(g\circ g)(x)=g(g(x))=g\left(\frac{x+1}{x+2}\right)=\frac{\frac{x+1}{x+2}+1}{\frac{x+1}{x+2}+2}=\frac{\frac{x+1+1(x+2)}{x+2}}{\frac{x+1+2(x+2)}{x+2}}=\frac{x+1+x+2}{x+1+2x+4}=\frac{2x+3}{3x+5}.$$

Since $g(x)$ is not defined for $x=-2$ and $g(g(x))$ is not defined for $x=-\frac{5}{3}$, the domain of $(g\circ g)(x)$ is $D=\left\{x\mid x\neq -2,-\frac{5}{3}\right\}$.

40. $f(x) = \sqrt{2x+3}$, $D = \{x \mid x \geq -\frac{3}{2}\}$; $g(x) = x^2 + 1$, $D = \mathbb{R}$.

$(f \circ g)(x) = f(x^2+1) = \sqrt{2(x^2+1)+3} = \sqrt{2x^2+5}$, $D = \mathbb{R}$.

$(g \circ f)(x) = g\left(\sqrt{2x+3}\right) = \left(\sqrt{2x+3}\right)^2 + 1 = (2x+3)+1 = 2x+4$, $D = \{x \mid x \geq -\frac{3}{2}\}$.

$(f \circ f)(x) = f\left(\sqrt{2x+3}\right) = \sqrt{2\left(\sqrt{2x+3}\right)+3} = \sqrt{2\sqrt{2x+3}+3}$, $D = \{x \mid x \geq -\frac{3}{2}\}$.

$(g \circ g)(x) = g(x^2+1) = (x^2+1)^2+1 = (x^4+2x^2+1)+1 = x^4+2x^2+2$, $D = \mathbb{R}$.

41. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x-1)) = f(2(x-1))$
$= 2(x-1)+1 = 2x-1$

42. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(1-x)) = f((1-x)^2)$
$= 2(1-x)^2 - 1 = 2x^2 - 4x + 1$

43. $(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x+3)) = f((x+3)^2+2)$
$= f(x^2+6x+11) = \sqrt{(x^2+6x+11)-1} = \sqrt{x^2+6x+10}$

44. $(f \circ g \circ h)(x) = f(g(h(x))) = f\left(g\left(\sqrt{x+3}\right)\right) = f\left(\cos\sqrt{x+3}\right) = \dfrac{2}{\cos\sqrt{x+3}+1}$

45. Let $g(x) = x^2+1$ and $f(x) = x^{10}$. Then $(f \circ g)(x) = f(g(x)) = (x^2+1)^{10} = F(x)$.

46. Let $g(x) = \sqrt{x}$ and $f(x) = \sin x$. Then $(f \circ g)(x) = f(g(x)) = \sin\left(\sqrt{x}\right) = F(x)$.

47. Let $g(x) = x^2$ and $f(x) = \dfrac{x}{x+4}$. Then $(f \circ g)(x) = f(g(x)) = \dfrac{x^2}{x^2+4} = G(x)$.

48. Let $g(x) = x+3$ and $f(x) = 1/x$. Then $(f \circ g)(x) = f(g(x)) = 1/(x+3) = G(x)$.

49. Let $g(t) = \cos t$ and $f(t) = \sqrt{t}$. Then $(f \circ g)(t) = f(g(t)) = \sqrt{\cos t} = u(t)$.

50. Let $g(t) = \tan t$ and $f(t) = \dfrac{t}{1+t}$. Then $(f \circ g)(t) = f(g(t)) = \dfrac{\tan t}{1+\tan t} = u(t)$.

51. Let $h(x) = x^2$, $g(x) = 3^x$, and $f(x) = 1-x$. Then $(f \circ g \circ h)(x) = 1 - 3^{x^2} = H(x)$.

52. Let $h(x) = \sqrt{x}$, $g(x) = x-1$, and $f(x) = \sqrt[3]{x}$. Then $(f \circ g \circ h)(x) = \sqrt[3]{\sqrt{x}-1} = H(x)$.

53. Let $h(x) = \sqrt{x}$, $g(x) = \sec x$, and $f(x) = x^4$. Then $(f \circ g \circ h)(x) = \left(\sec\sqrt{x}\right)^4 = \sec^4\left(\sqrt{x}\right) = H(x)$.

54. (a) $f(g(1)) = f(6) = 5$
(b) $g(f(1)) = g(3) = 2$
(c) $f(f(1)) = f(3) = 4$
(d) $g(g(1)) = g(6) = 3$
(e) $(g \circ f)(3) = g(f(3)) = g(4) = 1$
(f) $(f \circ g)(6) = f(g(6)) = f(3) = 4$

55. (a) $g(2) = 5$, because the point $(2,5)$ is on the graph of g. Thus, $f(g(2)) = f(5) = 4$, because the point $(5,4)$ is on the graph of f.

(b) $g(f(0)) = g(0) = 3$

(c) $(f \circ g)(0) = f(g(0)) = f(3) = 0$

(d) $(g \circ f)(6) = g(f(6)) = g(6)$. This value is not defined, because there is no point on the graph of g that has x-coordinate 6.

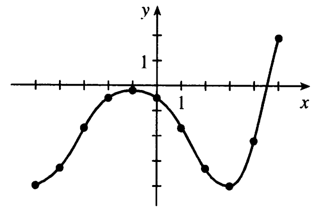
(e) $(g \circ g)(-2) = g(g(-2)) = g(1) = 4$

(f) $(f \circ f)(4) = f(f(4)) = f(2) = -2$

56. To find a particular value of $f(g(x))$, say for $x = 0$, we note from the graph that $g(0) \approx 2.8$ and $f(2.8) \approx -0.5$. Thus, $f(g(0)) \approx f(2.8) \approx -0.5$. The other values listed in the table were obtained in a similar fashion.

x	$g(x)$	$f(g(x))$
-5	-0.2	-4
-4	1.2	-3.3
-3	2.2	-1.7
-2	2.8	-0.5
-1	3	-0.2

x	$g(x)$	$f(g(x))$
0	2.8	-0.5
1	2.2	-1.7
2	1.2	-3.3
3	-0.2	-4
4	-1.9	-2.2
5	-4.1	1.9



57. (a) Using the relationship *distance* = *rate* · *time* with the radius r as the distance, we have $r(t) = 60t$.

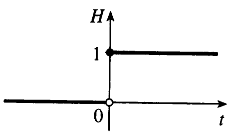
(b) $A = \pi r^2 \;\Rightarrow\; (A \circ r)(t) = A(r(t)) = \pi(60t)^2 = 3600\pi t^2$. This formula gives us the extent of the rippled area (in cm^2) at any time t.

58. (a) $d = rt \;\Rightarrow\; d(t) = 350t$

(b) There is a Pythagorean relationship involving the legs with lengths d and 1 and the hypotenuse with length s: $d^2 + 1^2 = s^2$. Thus, $s(d) = \sqrt{d^2+1}$.

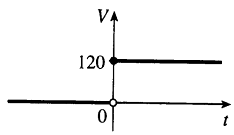
(c) $(s \circ d)(t) = s(d(t)) = s(350t) = \sqrt{(350t)^2+1}$

59. (a)



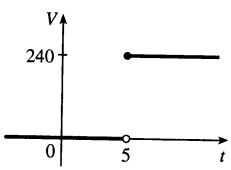
$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \ge 0 \end{cases}$$

(b)



$$V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 120 & \text{if } t \ge 0 \end{cases} \quad \text{so } V(t) = 120H(t).$$

(c)



Starting with the formula in part (b), we replace 120 with 240 to reflect the different voltage. Also, because we are starting 5 units to the right of $t = 0$, we replace t with $t - 5$. Thus, the formula is $V(t) = 240H(t-5)$.

60. (a) $R(t) = tH(t)$

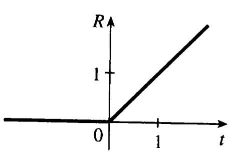
$$= \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } t \ge 0 \end{cases}$$

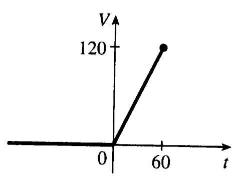
(b) $V(t) = \begin{cases} 0 & \text{if } t < 0 \\ 2t & \text{if } 0 \le t \le 60 \end{cases}$

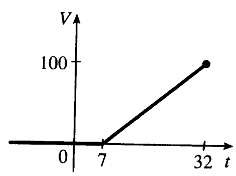
so $V(t) = 2tH(t)$, $t \le 60$.

(c) $V(t) = \begin{cases} 0 & \text{if } t < 7 \\ 4(t-7) & \text{if } 7 \le t \le 32 \end{cases}$

so $V(t) = 4(t-7)H(t-7)$, $t \le 32$.







61. (a) By examining the variable terms in g and h, we deduce that we must square g to get the terms $4x^2$ and $4x$ in h. If we let $f(x) = x^2 + c$, then $(f \circ g)(x) = f(g(x)) = f(2x+1) = (2x+1)^2 + c = 4x^2 + 4x + (1+c)$. Since $h(x) = 4x^2 + 4x + 7$, we must have $1 + c = 7$. So $c = 6$ and $f(x) = x^2 + 6$.

(b) We need a function g so that $f(g(x)) = 3(g(x)) + 5 = h(x)$. But $h(x) = 3x^2 + 3x + 2 = 3(x^2 + x) + 2 = 3(x^2 + x - 1) + 5$, so we see that $g(x) = x^2 + x - 1$.

62. We need a function g so that $g(f(x)) = g(x+4) = h(x) = 4x - 1 = 4(x+4) - 17$. So we see that the function g must be $g(x) = 4x - 17$.

63. We need to examine $h(-x)$.

$$h(-x) = (f \circ g)(-x) = f(g(-x)) = f(g(x)) \quad \text{[because } g \text{ is even]} \quad = h(x)$$

Because $h(-x) = h(x)$, h is an even function.

64. $h(-x) = f(g(-x)) = f(-g(x))$. At this point, we can't simplify the expression, so we might try to find a counterexample to show that h is not an odd function. Let $g(x) = x$, an odd function, and $f(x) = x^2 + x$. Then $h(x) = x^2 + x$, which is neither even nor odd.

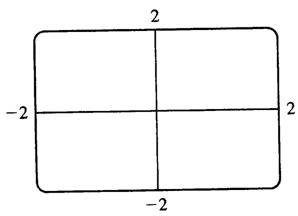
Now suppose f is an odd function. Then $f(-g(x)) = -f(g(x)) = -h(x)$. Hence, $h(-x) = -h(x)$, and so h is odd if both f and g are odd.

Now suppose f is an even function. Then $f(-g(x)) = f(g(x)) = h(x)$. Hence, $h(-x) = h(x)$, and so h is even if g is odd and f is even.

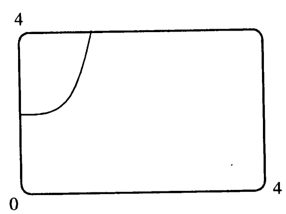
1.4 Graphing Calculators and Computers

1. $f(x) = x^4 + 2$

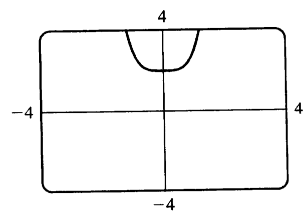
(a) $[-2, 2]$ by $[-2, 2]$



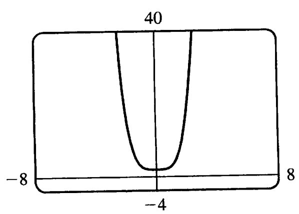
(b) $[0, 4]$ by $[0, 4]$



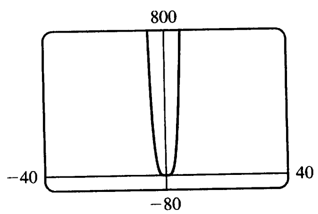
(c) $[-4, 4]$ by $[-4, 4]$



(d) $[-8, 8]$ by $[-4, 40]$



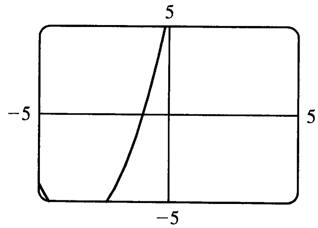
(e) $[-40, 40]$ by $[-80, 800]$



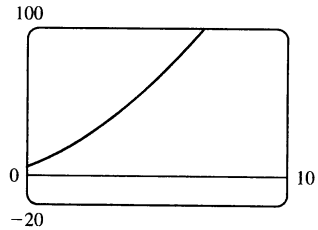
The most appropriate graph is produced in viewing rectangle (d).

2. $f(x) = x^2 + 7x + 6$

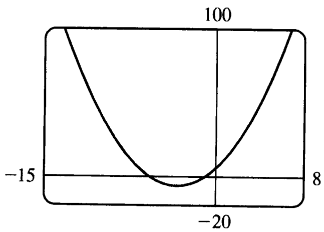
(a) $[-5, 5]$ by $[-5, 5]$



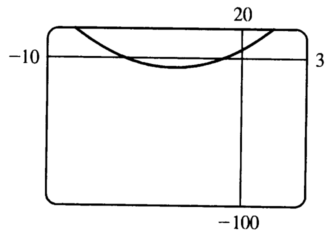
(b) $[0, 10]$ by $[-20, 100]$



(c) $[-15, 8]$ by $[-20, 100]$



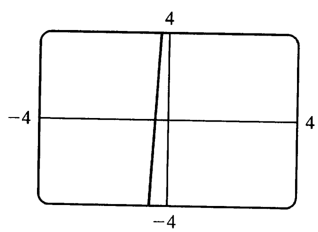
(d) $[-10, 3]$ by $[-100, 20]$



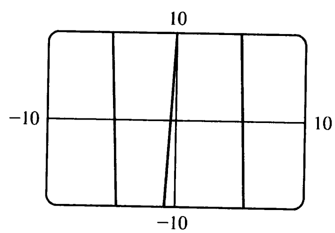
The most appropriate graph is produced in viewing rectangle (c).

3. $f(x) = 10 + 25x - x^3$

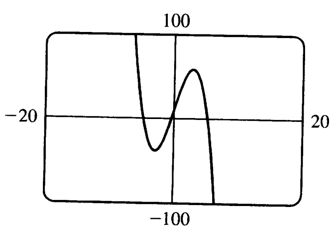
(a) $[-4, 4]$ by $[-4, 4]$



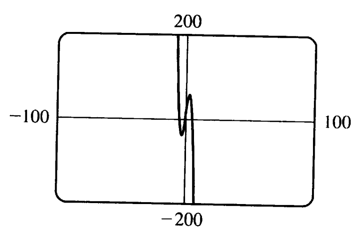
(b) $[-10, 10]$ by $[-10, 10]$



(c) $[-20, 20]$ by $[-100, 100]$



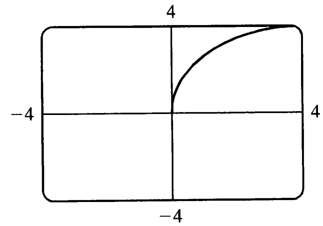
(d) $[-100, 100]$ by $[-200, 200]$



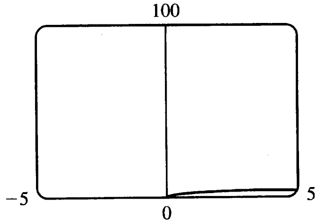
The most appropriate graph is produced in viewing rectangle (c) because the maximum and minimum points are fairly easy to see and estimate.

4. $f(x) = \sqrt{8x - x^2}$

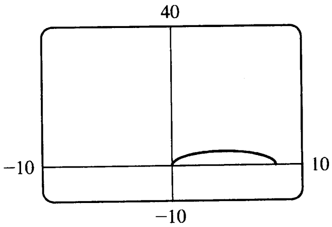
(a) $[-4, 4]$ by $[-4, 4]$



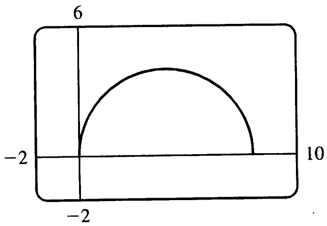
(b) $[-5, 5]$ by $[0, 100]$



(c) $[-10, 10]$ by $[-10, 40]$

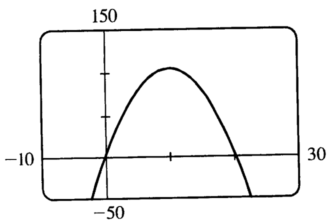


(d) $[-2, 10]$ by $[-2, 6]$

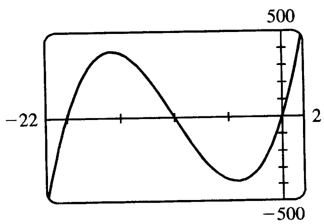


The most appropriate graph is produced in viewing rectangle (d).

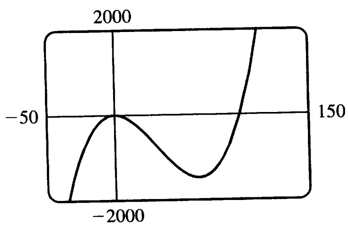
5. Since the graph of $f(x) = 5 + 20x - x^2$ is a parabola opening downward, an appropriate viewing rectangle should include the maximum point.



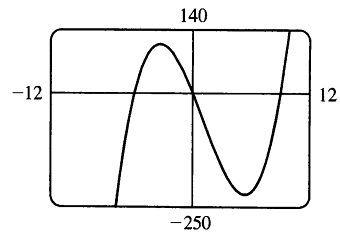
6. An appropriate viewing rectangle for $f(x) = x^3 + 30x^2 + 200x$ should include the high and low points.



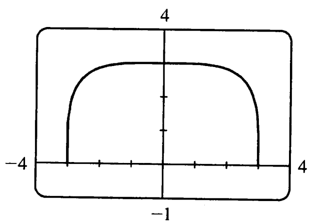
7. $f(x) = 0.01x^3 - x^2 + 5$. Graphing f in a standard viewing rectangle, $[-10, 10]$ by $[-10, 10]$, shows us what appears to be a parabola. But since this is a cubic polynomial, we know that a larger viewing rectangle will reveal a minimum point as well as the maximum point. After some trial and error, we choose the viewing rectangle $[-50, 150]$ by $[-2000, 2000]$.



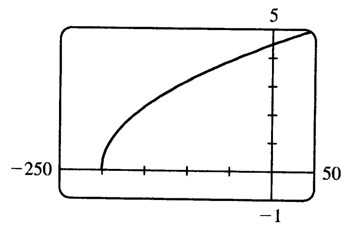
8. $f(x) = x(x+6)(x-9)$



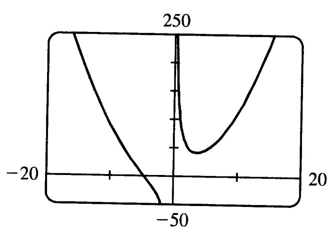
9. $f(x) = \sqrt[4]{81 - x^4}$ is defined when $81 - x^4 \geq 0 \quad \Leftrightarrow \quad x^4 \leq 81 \quad \Leftrightarrow \quad |x| \leq 3$, so the domain of f is $[-3, 3]$. Also $0 \leq \sqrt[4]{81 - x^4} \leq \sqrt[4]{81} = 3$, so the range is $[0, 3]$.



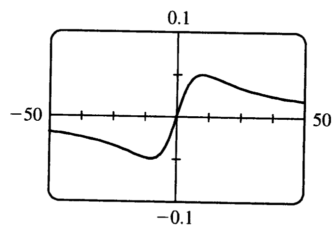
10. $f(x) = \sqrt{0.1x + 20}$ is defined when $0.1x + 20 \geq 0 \quad \Leftrightarrow \quad x \geq -200$, so the domain of f is $[-200, \infty)$.



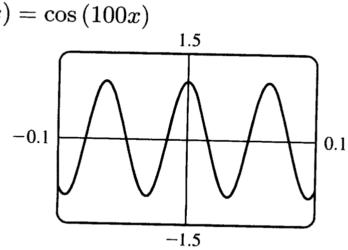
11. The graph of $f(x) = x^2 + (100/x)$ has a vertical asymptote of $x = 0$. As you zoom out, the graph of f looks more and more like that of $y = x^2$.



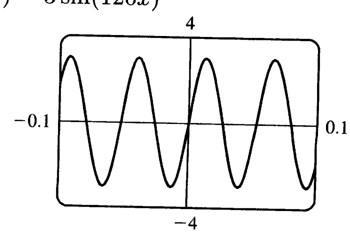
12. The graph of $f(x) = x/(x^2 + 100)$ is symmetric with respect to the origin.



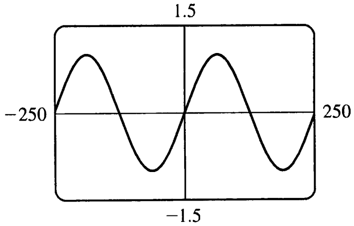
13. $f(x) = \cos(100x)$



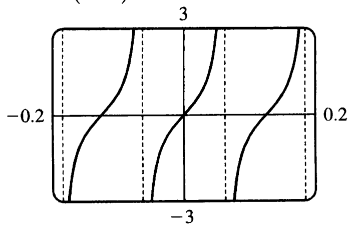
14. $f(x) = 3\sin(120x)$



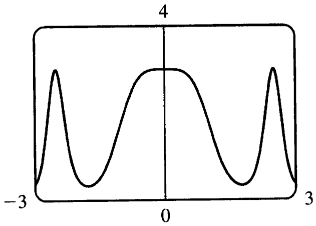
15. $f(x) = \sin(x/40)$

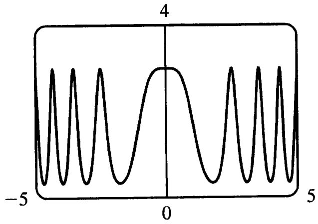


16. $f(x) = \tan(25x)$

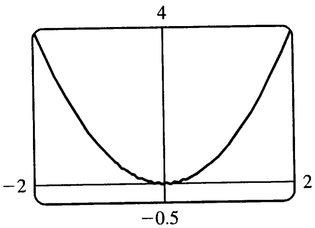


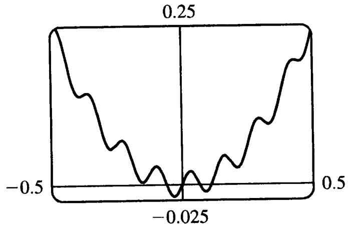
17. $y = 3^{\cos(x^2)}$





18. $y = x^2 + 0.02\sin(50x)$

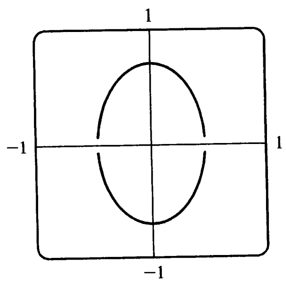




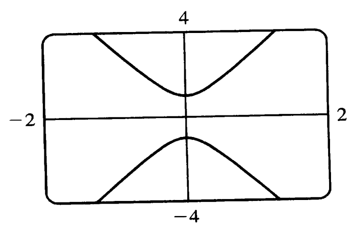
19. We must solve the given equation for y to obtain equations for the upper and lower halves of the ellipse.

$$4x^2 + 2y^2 = 1 \quad \Leftrightarrow \quad 2y^2 = 1 - 4x^2 \quad \Leftrightarrow \quad y^2 = \frac{1 - 4x^2}{2}$$

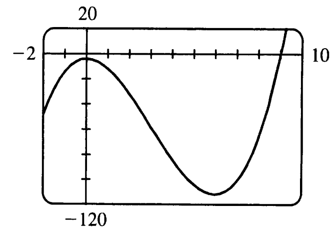
$$\Leftrightarrow \quad y = \pm\sqrt{\frac{1 - 4x^2}{2}}$$



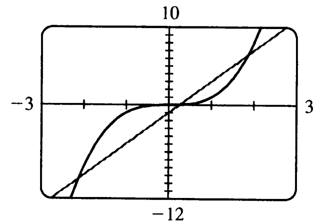
20. $y^2 - 9x^2 = 1 \quad \Leftrightarrow \quad y^2 = 1 + 9x^2 \quad \Leftrightarrow \quad y = \pm\sqrt{1 + 9x^2}$



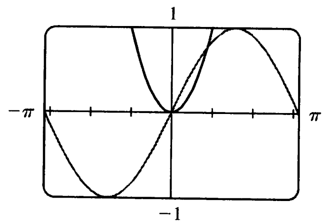
21. From the graph of $f(x) = x^3 - 9x^2 - 4$, we see that there is one solution of the equation $f(x) = 0$ and it is slightly larger than 9. By zooming in or using a root or zero feature, we obtain $x \approx 9.05$.



22. We see that the graphs of $f(x) = x^3$ and $g(x) = 4x - 1$ intersect three times. The x-coordinates of these points (which are the solutions of the equation) are approximately $-2.11, 0.25$, and 1.86. Alternatively, we could find these values by finding the zeros of $h(x) = x^3 - 4x + 1$.

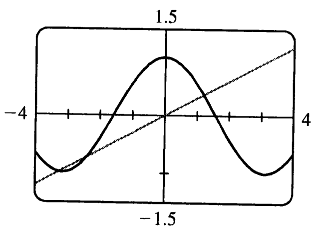


23. We see that the graphs of $f(x) = x^2$ and $g(x) = \sin x$ intersect twice. One solution is $x = 0$. The other solution of $f = g$ is the x-coordinate of the point of intersection in the first quadrant. Using an intersect feature or zooming in, we find this value to be approximately 0.88. Alternatively, we could find that value by finding the positive zero of $h(x) = x^2 - \sin x$.



Note: After producing the graph on a TI-83 Plus, we can find the approximate value 0.88 by using the following keystrokes: $\boxed{\text{2nd}}$ $\boxed{\text{CALC}}$ $\boxed{5}$ $\boxed{\text{ENTER}}$ $\boxed{\text{ENTER}}$ 1 $\boxed{\text{ENTER}}$. The "1" is just a guess for 0.88.

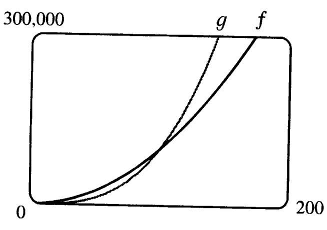
24. (a)



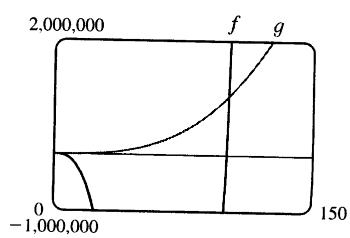
The x-coordinates of the three points of intersection are $x \approx -3.29$, -2.36 and 1.20.

(b) Using trial and error, we find that $m \approx 0.3365$. Note that m could also be negative.

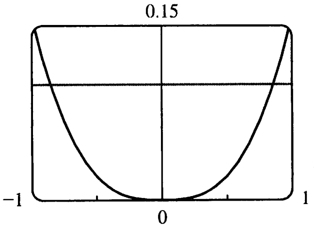
25. $g(x) = x^3/10$ is larger than $f(x) = 10x^2$ whenever $x > 100$.



26. $f(x) = x^4 - 100x^3$ is larger than $g(x) = x^3$ whenever $x > 101$.

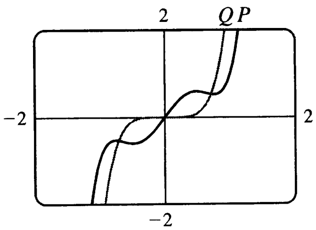


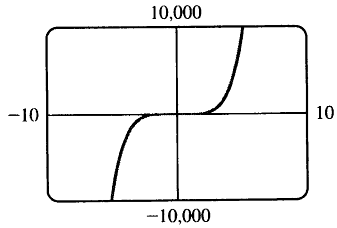
27.



We see from the graphs of $y = |\sin x - x|$ and $y = 0.1$ that there are two solutions to the equation $|\sin x - x| = 0.1$: $x \approx -0.85$ and $x \approx 0.85$. The condition $|\sin x - x| < 0.1$ holds for any x lying between these two values.

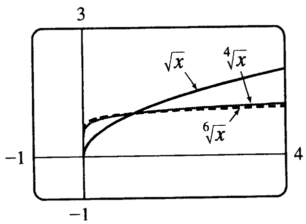
28.



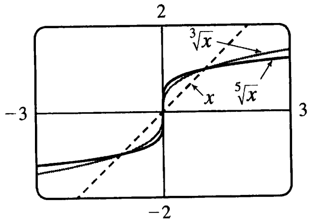


$P(x) = 3x^5 - 5x^3 + 2x$, $Q(x) = 3x^5$. These graphs are significantly different only in the region close to the origin. The larger a viewing rectangle one chooses, the more similar the two graphs look.

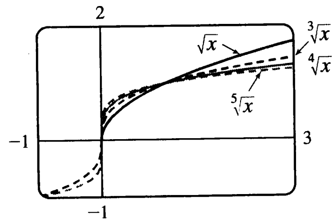
29. (a) The root functions $y = \sqrt{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[6]{x}$



(b) The root functions $y = x$, $y = \sqrt[3]{x}$ and $y = \sqrt[5]{x}$



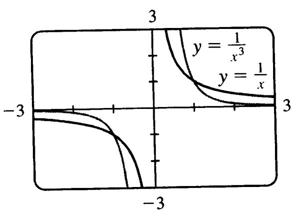
(c) The root functions $y = \sqrt{x}$, $y = \sqrt[3]{x}$, $y = \sqrt[4]{x}$ and $y = \sqrt[5]{x}$



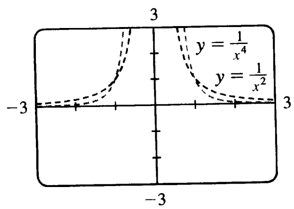
(d) • For any n, the nth root of 0 is 0 and the nth root of 1 is 1; that is, all nth root functions pass through the points $(0, 0)$ and $(1, 1)$.

- For odd n, the domain of the nth root function is $\mathbb{R}$, while for even n, it is $\{x \in \mathbb{R} \mid x \geq 0\}$.
- Graphs of even root functions look similar to that of $\sqrt{x}$, while those of odd root functions resemble that of $\sqrt[3]{x}$.
- As n increases, the graph of $\sqrt[n]{x}$ becomes steeper near 0 and flatter for $x > 1$.

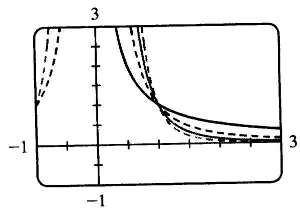
30. (a) The functions $y = 1/x$ and $y = 1/x^3$



(b) The functions $y = 1/x^2$ and $y = 1/x^4$



(c) The functions $y = 1/x$, $y = 1/x^2$, $y = 1/x^3$ and $y = 1/x^4$



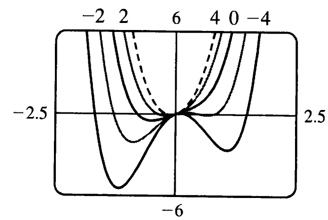
(d) • The graphs of all functions of the form $y = 1/x^n$ pass through the point $(1, 1)$.

• If n is even, the graph of the function is entirely above the x-axis. The graphs of $1/x^n$ for n even are similar to one another.

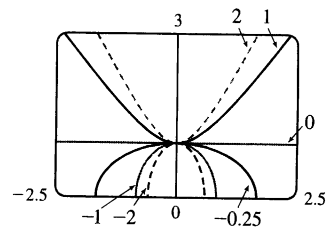
• If n is odd, the function is positive for positive x and negative for negative x. The graphs of $1/x^n$ for n odd are similar to one another.

• As n increases, the graphs of $1/x^n$ approach 0 faster as $x \to \infty$.

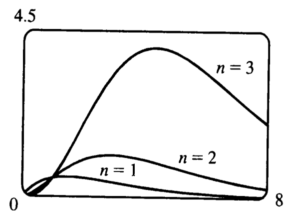
31. $f(x) = x^4 + cx^2 + x$. If $c < 0$, there are three humps: two minimum points and a maximum point. These humps get flatter as c increases, until at $c = 0$ two of the humps disappear and there is only one minimum point. This single hump then moves to the right and approaches the origin as c increases.

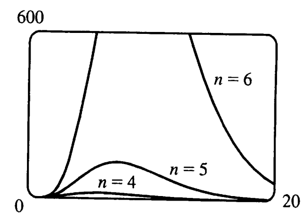


32. $f(x) = \sqrt{1 + cx^2}$. If $c < 0$, the function is only defined on $\left[-1/\sqrt{-c}\,, 1/\sqrt{-c}\,\right]$, and its graph is the top half of an ellipse. If $c = 0$, the graph is the line $y = 1$. If $c > 0$, the graph is the top half of a hyperbola. As c approaches 0, these curves become flatter and approach the line $y = 1$.

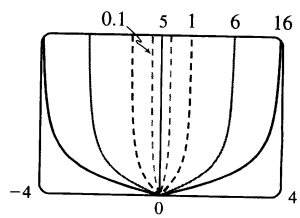


33. $y = x^n 2^{-x}$. As n increases, the maximum of the function moves further from the origin, and gets larger. Note, however, that regardless of n, the function approaches 0 as $x \to \infty$.



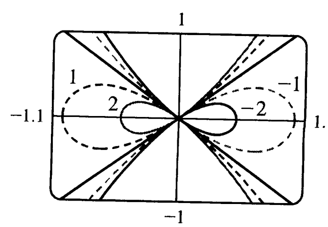


34. $y = \dfrac{|x|}{\sqrt{c - x^2}}$. The "bullet" becomes broader as c increases.



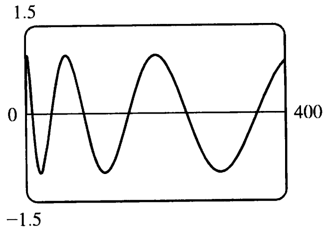
35. $y^2 = cx^3 + x^2$

If $c < 0$, the loop is to the right of the origin, and if c is positive, it is to the left. In both cases, the closer c is to 0, the larger the loop is. (In the limiting case, $c = 0$, the loop is "infinite", that is, it doesn't close.) Also, the larger $|c|$ is, the steeper the slope is on the loopless side of the origin.



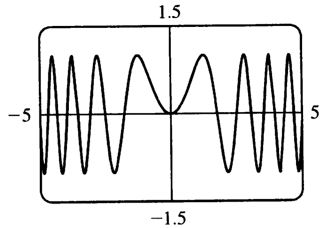
36. (a) $y = \sin(\sqrt{x})$

This function is not periodic; it oscillates less frequently as x increases.



(b) $y = \sin(x^2)$

This function oscillates more frequently as $|x|$ increases. Note also that this function is even, whereas $\sin x$ is odd.



37. The graphing window is 95 pixels wide and we want to start with $x = 0$ and end with $x = 2\pi$. Since there are 94 "gaps" between pixels, the distance between pixels is $\frac{2\pi - 0}{94}$. Thus, the x-values that the calculator actually plots are $x = 0 + \frac{2\pi}{94} \cdot n$, where $n = 0, 1, 2, \ldots, 93, 94$. For $y = \sin 2x$, the actual points plotted by the calculator are $\left(\frac{2\pi}{94} \cdot n, \sin\left(2 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. For $y = \sin 96x$, the points plotted are $\left(\frac{2\pi}{94} \cdot n, \sin\left(96 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. But

$$\begin{aligned}\sin\left(96 \cdot \tfrac{2\pi}{94} \cdot n\right) &= \sin\left(94 \cdot \tfrac{2\pi}{94} \cdot n + 2 \cdot \tfrac{2\pi}{94} \cdot n\right) = \sin\left(2\pi n + 2 \cdot \tfrac{2\pi}{94} \cdot n\right) \\ &= \sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) \quad \text{[by periodicity of sine]}, \quad n = 0, 1, \ldots, 94\end{aligned}$$

So the y-values, and hence the points, plotted for $y = \sin 96x$ are identical to those plotted for $y = \sin 2x$.

Note: Try graphing $y = \sin 94x$. Can you see why all the y-values are zero?

38. As in Exercise 37, we know that the points being plotted for $y = \sin 45x$ are $\left(\frac{2\pi}{94} \cdot n, \sin\left(45 \cdot \frac{2\pi}{94} \cdot n\right)\right)$ for $n = 0, 1, \ldots, 94$. But

$$\begin{aligned}\sin\left(45 \cdot \tfrac{2\pi}{94} \cdot n\right) &= \sin\left(47 \cdot \tfrac{2\pi}{94} \cdot n - 2 \cdot \tfrac{2\pi}{94} \cdot n\right) = \sin\left(n\pi - 2 \cdot \tfrac{2\pi}{94} \cdot n\right) \\ &= \sin(n\pi)\cos\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) - \cos(n\pi)\sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) \quad \text{[Subtraction formula for the sine]} \\ &= 0 \cdot \cos\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) - (\pm 1)\sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right) = \pm\sin\left(2 \cdot \tfrac{2\pi}{94} \cdot n\right), \quad n = 0, 1, \ldots, 94\end{aligned}$$

So the y-values, and hence the points, plotted for $y = \sin 45x$ lie on either $y = \sin 2x$ or $y = -\sin 2x$.

1.5 Exponential Functions

1. (a) $f(x) = a^x$, $a > 0$

(b) $\mathbb{R}$

(c) $(0, \infty)$

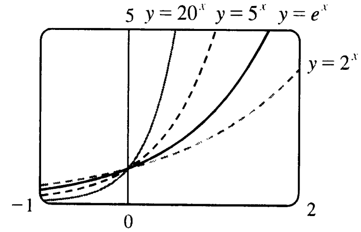
(d) See Figures 6(c), 6(b), and 6(a), respectively.

2. (a) The number e is the value of a such that the slope of the tangent line at $x = 0$ on the graph of $y = a^x$ is exactly 1.

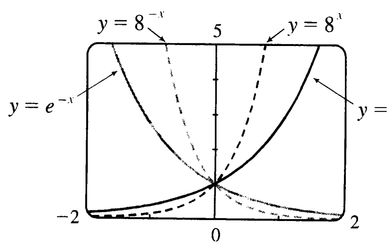
(b) $e \approx 2.71828$

(c) $f(x) = e^x$

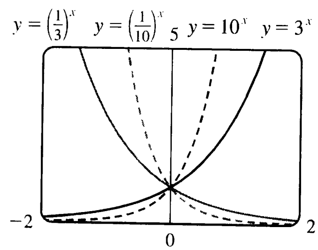
3. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.



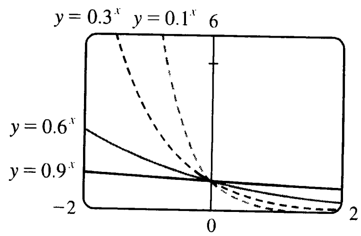
4. The graph of e^{-x} is the reflection of the graph of e^x about the y-axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y-axis. The graph of 8^x increases more quickly than that of e^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



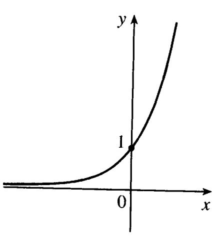
5. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



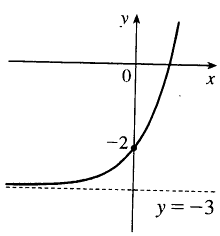
6. Each of the graphs approaches ∞ as $x \to -\infty$, and each approaches 0 as $x \to \infty$. The smaller the base, the faster the function grows as $x \to -\infty$, and the faster it approaches 0 as $x \to \infty$.



7. We start with the graph of $y = 4^x$ (Figure 3) and then shift 3 units downward. This shift doesn't affect the domain, but the range of $y = 4^x - 3$ is $(-3, \infty)$. There is a horizontal asymptote of $y = -3$.

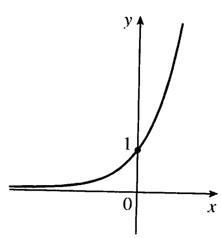


$y = 4^x$

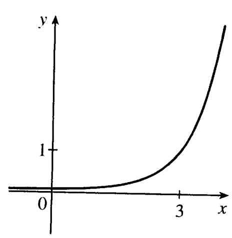


$y = 4^x - 3$

8. We start with the graph of $y = 4^x$ (Figure 3) and then shift 3 units to the right. There is a horizontal asymptote of $y = 0$.

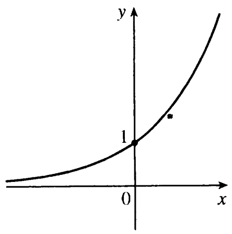


$y = 4^x$

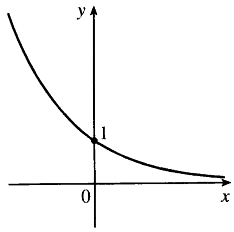


$y = 4^{x-3}$

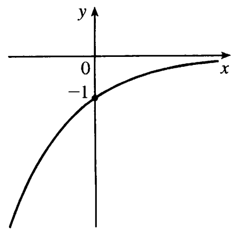
9. We start with the graph of $y = 2^x$ (Figure 2), reflect it about the y-axis, and then about the x-axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -2^{-x}$. In each graph, $y = 0$ is the horizontal asymptote.



$y = 2^x$

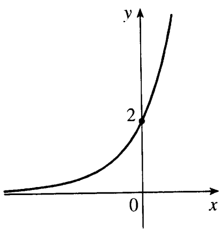


$y = 2^{-x}$

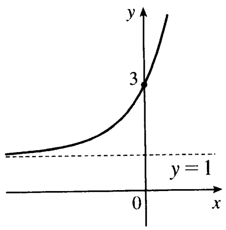


$y = -2^{-x}$

10. We start with the graph of $y = e^x$ (Figure 13), vertically stretch by a factor of 2, and then shift 1 unit upward. There is a horizontal asymptote of $y = 1$.

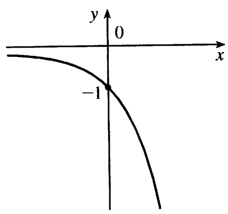


$y = 2e^x$

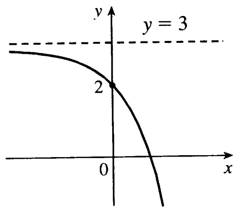


$y = 1 + 2e^x$

11. We start with the graph of $y = e^x$ (Figure 13), reflect it about the x-axis, and then shift 3 units upward. Note the horizontal asymptote of $y = 3$.

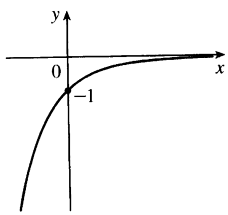


$y = -e^x$

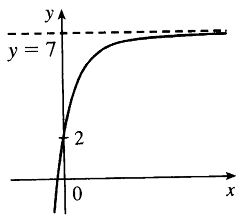


$y = 3 - e^x$

12. We start with the graph of $y = e^x$ (Figure 13), reflect it about the y-axis, and then about the x-axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -e^{-x}$. Now shift this graph 1 unit upward, vertically stretch by a factor of 5, and then shift 2 units upward.



$y = -e^{-x}$



$y = 2 + 5(1 - e^{-x})$

13. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

14. (a) This reflection consists of first reflecting the graph about the x-axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.

(b) This reflection consists of first reflecting the graph about the y-axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

15. (a) The denominator $1 + e^x$ is never equal to zero because $e^x > 0$, so the domain of $f(x) = 1/(1 + e^x)$ is $\mathbb{R}$.

(b) $1 - e^x = 0 \quad \Leftrightarrow \quad e^x = 1 \quad \Leftrightarrow \quad x = 0$, so the domain of $f(x) = 1/(1 - e^x)$ is $(-\infty, 0) \cup (0, \infty)$.

16. (a) The sine and exponential functions have domain $\mathbb{R}$, so $g(t) = \sin(e^{-t})$ also has domain $\mathbb{R}$.

(b) The function $g(t) = \sqrt{1 - 2^t}$ has domain $\{t \mid 1 - 2^t \geq 0\} = \{t \mid 2^t \leq 1\} = \{t \mid t \leq 0\} = (-\infty, 0]$.

17. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Ca^1 \quad \left[C = \frac{6}{a}\right]$ and $24 = Ca^3 \quad \Rightarrow \quad 24 = \left(\dfrac{6}{a}\right)a^3 \quad \Rightarrow$ $4 = a^2 \quad \Rightarrow \quad a = 2 \quad$ [since $a > 0$] and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

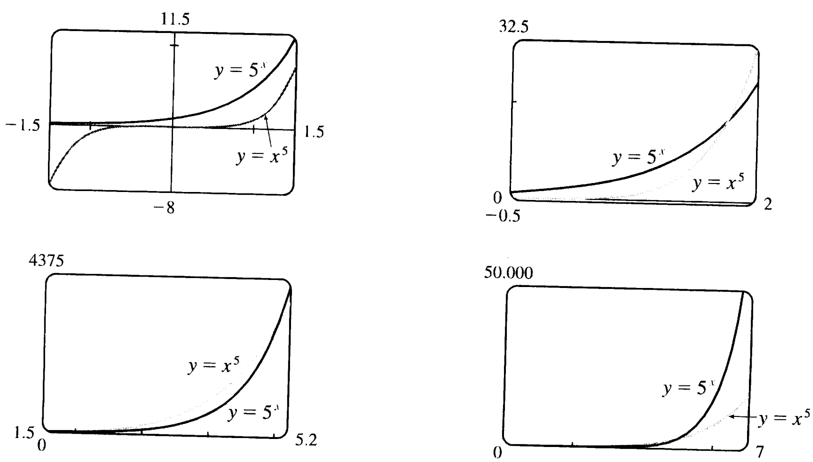
18. Given the y-intercept $(0, 2)$, we have $y = Ca^x = 2a^x$. Using the point $\left(2, \frac{2}{9}\right)$ gives us $\frac{2}{9} = 2a^2 \quad \Rightarrow \quad \frac{1}{9} = a^2$ $\Rightarrow \quad a = \frac{1}{3} \quad$ [since $a > 0$]. The function is $f(x) = 2\left(\frac{1}{3}\right)^x$ or $f(x) = 2(3)^{-x}$.

19. If $f(x) = 5^x$, then $\dfrac{f(x+h) - f(x)}{h} = \dfrac{5^{x+h} - 5^x}{h} = \dfrac{5^x 5^h - 5^x}{h} = \dfrac{5^x(5^h - 1)}{h} = 5^x\left(\dfrac{5^h - 1}{h}\right)$.

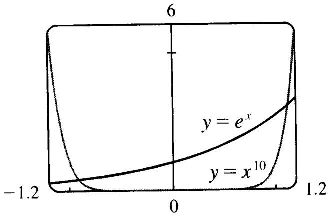
20. Suppose the month is February. Your payment on the 28th day would be $2^{28-1} = 2^{27} = 134{,}217{,}728$ cents, or \$1,342,177.28. Clearly, the second method of payment results in a larger amount for any month.

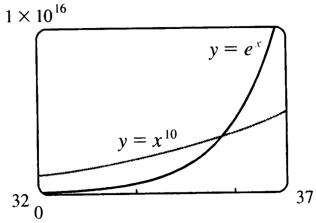
21. $2 \text{ ft} = 24 \text{ in}$, $f(24) = 24^2 \text{ in} = 576 \text{ in} = 48 \text{ ft}$. $g(24) = 2^{24} \text{ in} = 2^{24}/(12 \cdot 5280) \text{ mi} \approx 265 \text{ mi}$

22. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.

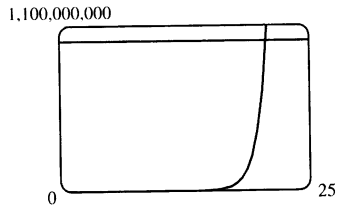


23. The graph of g finally surpasses that of f at $x \approx 35.8$.





24. We graph $y = e^x$ and $y = 1{,}000{,}000{,}000$ and determine where $e^x = 1 \times 10^9$. This seems to be true at $x \approx 20.723$, so $e^x > 1 \times 10^9$ for $x > 20.723$.



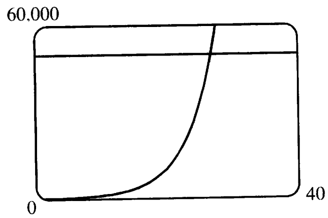
25. (a) Fifteen hours represents 5 doubling periods (one doubling period is three hours).

$100 \cdot 2^5 = 3200$

(b) In t hours, there will be $t/3$ doubling periods. The initial population is 100, so the population y at time t is $y = 100 \cdot 2^{t/3}$.

(c) $t = 20 \quad \Rightarrow \quad y = 100 \cdot 2^{20/3} \approx 10{,}159$

(d) We graph $y_1 = 100 \cdot 2^{x/3}$ and $y_2 = 50{,}000$. The two curves intersect at $x \approx 26.9$, so the population reaches 50,000 in about 26.9 hours.



26. (a) Sixty hours represents 4 half-life periods.

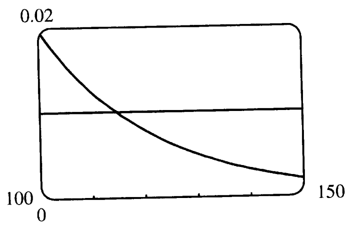
$2 \cdot \left(\frac{1}{2}\right)^4 = \frac{1}{8}$ g

(b) In t hours, there will be $t/15$ half-life periods. The initial mass is 2 g, so the mass y at time t is $y = 2 \cdot \left(\frac{1}{2}\right)^{t/15}$.

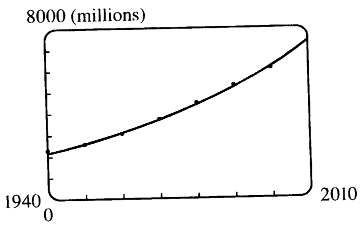
(c) 4 days $= 4 \cdot 24 = 96$ hours. $t = 96 \quad \Rightarrow$

$y = 2 \cdot \left(\frac{1}{2}\right)^{96/15} \approx 0.024$ g

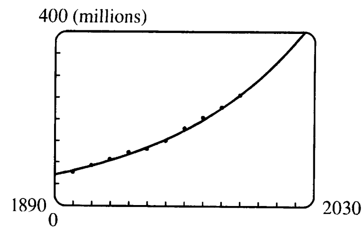
(d) $y = 0.01 \quad \Rightarrow \quad t \approx 114.7$ hours



27. An exponential model is $y = ab^t$, where $a = 3.154832569 \times 10^{-12}$ and $b = 1.017764706$. This model gives $y(1993) \approx 5498$ million and $y(2010) \approx 7417$ million.



28. An exponential model is $y = ab^t$, where
$a = 1.9976760197589 \times 10^{-9}$ and $b = 1.0129334321697$.
This model gives $y(1925) \approx 111$ million,
$y(2010) \approx 330$ million, and $y(2020) \approx 375$ million.



1.6 Inverse Functions and Logarithms

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

2. (a) $f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$ for any y in B. The domain of f^{-1} is B and the range of f^{-1} is A.

(b) See the steps in (5).

(c) Reflect the graph of f about the line $y = x$.

3. f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.

4. f is one-to-one since for any two different domain values, there are different range values.

5. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.

6. The horizontal line $y = 0$ (the x-axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.

7. The horizontal line $y = 0$ (the x-axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.

8. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.

9. The graph of $f(x) = \frac{1}{2}(x + 5)$ is a line with slope $\frac{1}{2}$. It passes the Horizontal Line Test, so f is one-to-one.
Algebraic solution: If $x_1 \neq x_2$, then $x_1 + 5 \neq x_2 + 5 \quad \Rightarrow \quad \frac{1}{2}(x_1 + 5) \neq \frac{1}{2}(x_2 + 5) \quad \Rightarrow \quad f(x_1) \neq f(x_2)$, so f is one-to-one.

10. The graph of $f(x) = 1 + 4x - x^2$ is a parabola with axis of symmetry $x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$. Pick any x-values equidistant from 2 to find two equal function values. For example, $f(1) = 4$ and $f(3) = 4$, so f is not 1-1.

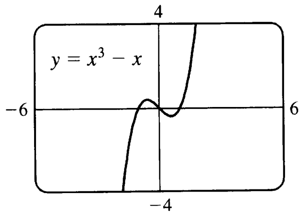
11. $g(x) = |x| \quad \Rightarrow \quad g(-1) = 1 = g(1)$, so g is not one-to-one.

12. $x_1 \neq x_2 \quad \Rightarrow \quad \sqrt{x_1} \neq \sqrt{x_2} \quad \Rightarrow \quad g(x_1) \neq g(x_2)$, so g is 1-1.

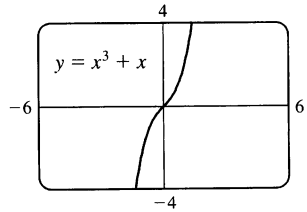
13. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

14. f is not 1-1 because eventually we all stop growing and therefore, there are two times at which we have the same height.

15. f does not pass the Horizontal Line Test, so f is not 1-1.



16. f passes the Horizontal Line Test, so f is 1-1.



17. Since $f(2) = 9$ and f is 1-1, we know that $f^{-1}(9) = 2$. Remember, if the point $(2, 9)$ is on the graph of f, then the point $(9, 2)$ is on the graph of f^{-1}.

18. (a) First, we must determine x such that $f(x) = 3$. By inspection, we see that if $x = 0$, then $f(x) = 3$. Since f is 1-1 (f is an increasing function), it has an inverse, and $f^{-1}(3) = 0$.

(b) By the second cancellation equation in (4), we have $f\left(f^{-1}(5)\right) = 5$.

19. First, we must determine x such that $g(x) = 4$. By inspection, we see that if $x = 0$, then $g(x) = 4$. Since g is 1-1 (g is an increasing function), it has an inverse, and $g^{-1}(4) = 0$.

20. (a) f is 1-1 because it passes the Horizontal Line Test.

(b) Domain of $f = [-3, 3] =$ Range of f^{-1}. Range of $f = [-2, 2] =$ Domain of f^{-1}.

(c) Since $f(-2) = 1$, $f^{-1}(1) = -2$.

21. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \quad \Rightarrow \quad F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \quad \Rightarrow$ $\frac{9}{5}C + 32 \geq -459.67 \quad \Rightarrow \quad \frac{9}{5}C \geq -491.67 \quad \Rightarrow \quad C \geq -273.15$, the domain of the inverse function.

22. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} \quad \Rightarrow \quad 1 - \dfrac{v^2}{c^2} = \dfrac{m_0^2}{m^2} \quad \Rightarrow \quad \dfrac{v^2}{c^2} = 1 - \dfrac{m_0^2}{m^2} \quad \Rightarrow \quad v^2 = c^2\left(1 - \dfrac{m_0^2}{m^2}\right) \quad \Rightarrow$

$v = c\sqrt{1 - \dfrac{m_0^2}{m^2}}$. This formula gives us the speed v of the particle in terms of its mass m, that is, $v = f^{-1}(m)$.

23. $f(x) = \sqrt{10 - 3x} \quad \Rightarrow \quad y = \sqrt{10 - 3x} \quad (y \geq 0) \quad \Rightarrow \quad y^2 = 10 - 3x \quad \Rightarrow \quad 3x = 10 - y^2 \quad \Rightarrow$ $x = -\frac{1}{3}y^2 + \frac{10}{3}$. Interchange x and y: $y = -\frac{1}{3}x^2 + \frac{10}{3}$. So $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{10}{3}$. Note that the domain of f^{-1} is $x \geq 0$.

24. $f(x) = \dfrac{4x - 1}{2x + 3} \quad \Rightarrow \quad y = \dfrac{4x - 1}{2x + 3} \quad \Rightarrow \quad y(2x + 3) = 4x - 1 \quad \Rightarrow \quad 2xy + 3y = 4x - 1 \quad \Rightarrow$

$3y + 1 = 4x - 2xy \quad \Rightarrow \quad 3y + 1 = (4 - 2y)x \quad \Rightarrow \quad x = \dfrac{3y + 1}{4 - 2y}$. Interchange x and y: $y = \dfrac{3x + 1}{4 - 2x}$.

So $f^{-1}(x) = \dfrac{3x + 1}{4 - 2x}$.

25. $f(x) = e^{x^3} \quad \Rightarrow \quad y = e^{x^3} \quad \Rightarrow \quad \ln y = x^3 \quad \Rightarrow \quad x = \sqrt[3]{\ln y}$. Interchange x and y: $y = \sqrt[3]{\ln x}$. So $f^{-1}(x) = \sqrt[3]{\ln x}$.

26. $y = f(x) = 2x^3 + 3 \quad \Rightarrow \quad y - 3 = 2x^3 \quad \Rightarrow \quad \dfrac{y - 3}{2} = x^3 \quad \Rightarrow \quad x = \sqrt[3]{\dfrac{y - 3}{2}}$.

Interchange x and y: $y = \sqrt[3]{\dfrac{x - 3}{2}}$. So $f^{-1}(x) = \sqrt[3]{\dfrac{x - 3}{2}}$.

27. $y = \ln(x+3) \;\Rightarrow\; x + 3 = e^y \;\Rightarrow\; x = e^y - 3$. Interchange x and y: $y = e^x - 3$. So $f^{-1}(x) = e^x - 3$.

28. $y = \dfrac{1+e^x}{1-e^x} \;\Rightarrow\; y - ye^x = 1 + e^x \;\Rightarrow\; y - 1 = ye^x + e^x \;\Rightarrow\; y - 1 = e^x(y+1) \;\Rightarrow$

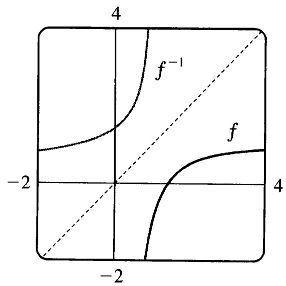
$e^x = \dfrac{y-1}{y+1} \;\Rightarrow\; x = \ln\!\left(\dfrac{y-1}{y+1}\right)$. Interchange x and y: $y = \ln\!\left(\dfrac{x-1}{x+1}\right)$. So $f^{-1}(x) = \ln\!\left(\dfrac{x-1}{x+1}\right)$.

Note that the domain of f^{-1} is $|x| > 1$.

29. $y = f(x) = 1 - \dfrac{2}{x^2} \;\Rightarrow\; 1 - y = \dfrac{2}{x^2} \;\Rightarrow\; x^2 = \dfrac{2}{1-y} \;\Rightarrow$

$x = \sqrt{\dfrac{2}{1-y}}$, since $x > 0$. Interchange x and y: $y = \sqrt{\dfrac{2}{1-x}}$.

So $f^{-1}(x) = \sqrt{\dfrac{2}{1-x}}$.

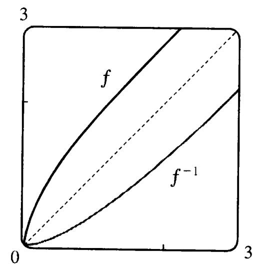


30. $y = f(x) = \sqrt{x^2 + 2x}$, $x > 0 \;\Rightarrow\; y > 0$ and $y^2 = x^2 + 2x \;\Rightarrow$

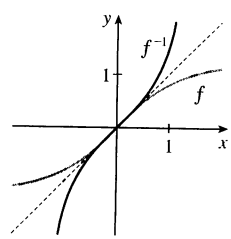
$x^2 + 2x - y^2 = 0$. Now we use the quadratic formula:

$x = \dfrac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-y^2)}}{2 \cdot 1} = -1 \pm \sqrt{1 + y^2}$. But $x > 0$, so the negative root is inadmissible. Interchange x and y: $y = -1 + \sqrt{1 + x^2}$.

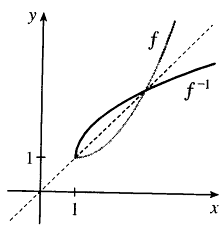
So $f^{-1}(x) = -1 + \sqrt{1 + x^2}$, $x > 0$.

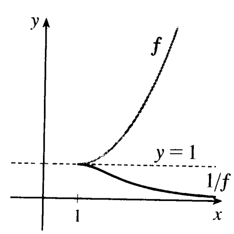


31. The function f is one-to-one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y = x$.



32. The function f is one-to-one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y = x$. For the graph of $1/f$, the y-coordinates are simply the reciprocals of f. For example, if $f(5) = 9$, then $1/f(5) = \frac{1}{9}$. If we draw the horizontal line $y = 1$, we see that the only place where the graphs intersect is on that line.





33. (a) It is defined as the inverse of the exponential function with base a, that is, $\log_a x = y \;\Leftrightarrow\; a^y = x$.

(b) $(0, \infty)$

(c) $\mathbb{R}$

(d) See Figure 11.

34. (a) The natural logarithm is the logarithm with base e, denoted $\ln x$.

(b) The common logarithm is the logarithm with base 10, denoted $\log x$.

(c) See Figure 13.

35. (a) $\log_2 64 = 6$ since $2^6 = 64$. (b) $\log_6 \frac{1}{36} = -2$ since $6^{-2} = \frac{1}{36}$.

36. (a) $\log_8 2 = \frac{1}{3}$ since $8^{1/3} = 2$. (b) $\ln e^{\sqrt{2}} = \sqrt{2}$

37. (a) $\log_{10} 1.25 + \log_{10} 80 = \log_{10}(1.25 \cdot 80) = \log_{10} 100 = \log_{10} 10^2 = 2$

(b) $\log_5 10 + \log_5 20 - 3\log_5 2 = \log_5(10 \cdot 20) - \log_5 2^3 = \log_5 \frac{200}{8} = \log_5 25 = \log_5 5^2 = 2$

38. (a) $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 15} = 15$ [*Or:* $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 3} \cdot 2^{\log_2 5} = 3 \cdot 5 = 15$]

(b) $e^{3\ln 2} = e^{\ln(2^3)} = e^{\ln 8} = 8$ [*Or:* $e^{3\ln 2} = \left(e^{\ln 2}\right)^3 = 2^3 = 8$]

39. $2\ln 4 - \ln 2 = \ln 4^2 - \ln 2 = \ln 16 - \ln 2 = \ln \frac{16}{2} = \ln 8$

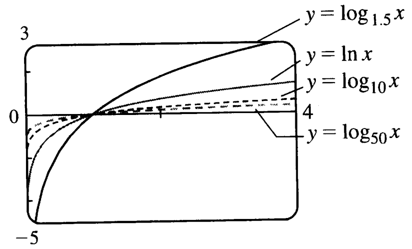
40. $\ln x + a\ln y - b\ln z = \ln x + \ln y^a - \ln z^b = \ln(x \cdot y^a) - \ln z^b = \ln(xy^a/z^b)$

41. $\ln(1+x^2) + \frac{1}{2}\ln x - \ln\sin x = \ln(1+x^2) + \ln x^{1/2} - \ln\sin x = \ln[(1+x^2)\sqrt{x}\,] - \ln\sin x = \ln\dfrac{(1+x^2)\sqrt{x}}{\sin x}$

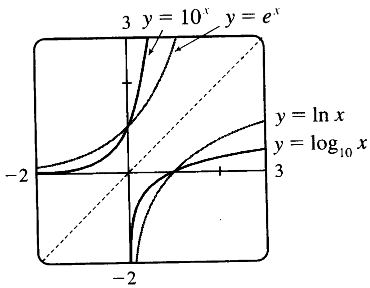
42. (a) $\log_{12} 10 = \dfrac{\ln 10}{\ln 12} \approx 0.926628$

(b) $\log_2 8.4 = \dfrac{\ln 8.4}{\ln 2} \approx 3.070389$

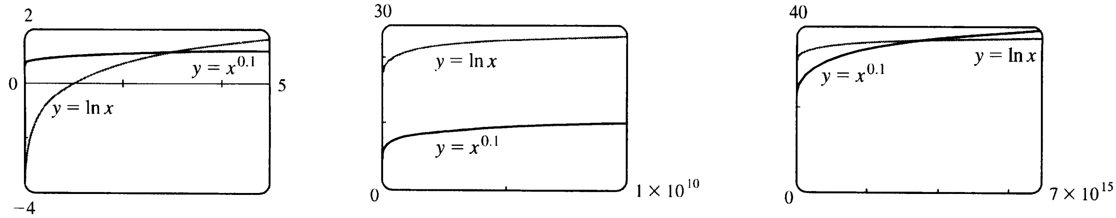
43. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.



44. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x. Also note that $\log_{10} x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



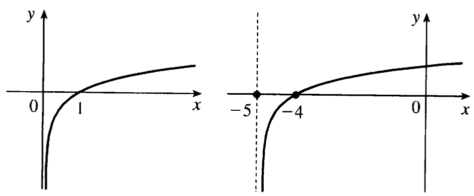
45. 3 ft = 36 in, so we need x such that $\log_2 x = 36 \Leftrightarrow x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is $68{,}719{,}476{,}736 \text{ in} \cdot \dfrac{1 \text{ ft}}{12 \text{ in}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7$ mi.

46. 

From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

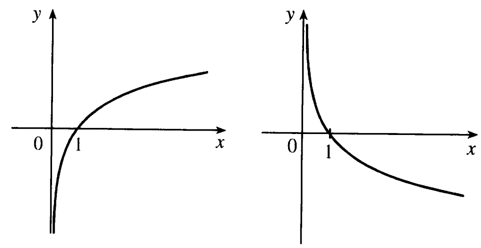
47. (a) Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x + 5)$. Note the vertical asymptote of $x = -5$.

$y = \log_{10} x$ $\qquad$ $y = \log_{10}(x + 5)$



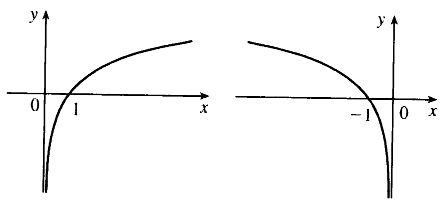
(b) Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.

$y = \ln x$ $\qquad$ $y = -\ln x$



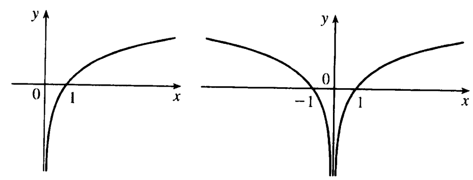
48. (a) Reflect the graph of $y = \ln x$ about the y-axis to obtain the graph of $y = \ln(-x)$.

$y = \ln x$ $\qquad$ $y = \ln(-x)$



(b) Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln|x|$ is that reflection in addition to the original portion.

$y = \ln x$ $\qquad$ $y = \ln|x|$



49. (a) $2\ln x = 1 \;\Rightarrow\; \ln x = \frac{1}{2} \;\Rightarrow\; x = e^{1/2} = \sqrt{e}$

(b) $e^{-x} = 5 \;\Rightarrow\; -x = \ln 5 \;\Rightarrow\; x = -\ln 5$

50. (a) $e^{2x+3} - 7 = 0 \;\Rightarrow\; e^{2x+3} = 7 \;\Rightarrow\; 2x + 3 = \ln 7 \;\Rightarrow\; 2x = \ln 7 - 3 \;\Rightarrow\; x = \frac{1}{2}(\ln 7 - 3)$

(b) $\ln(5 - 2x) = -3 \;\Rightarrow\; 5 - 2x = e^{-3} \;\Rightarrow\; 2x = 5 - e^{-3} \;\Rightarrow\; x = \frac{1}{2}(5 - e^{-3})$

51. (a) $2^{x-5} = 3 \;\Leftrightarrow\; \log_2 3 = x - 5 \;\Leftrightarrow\; x = 5 + \log_2 3$.

Or: $2^{x-5} = 3 \;\Leftrightarrow\; \ln(2^{x-5}) = \ln 3 \;\Leftrightarrow\; (x - 5)\ln 2 = \ln 3 \;\Leftrightarrow\; x - 5 = \dfrac{\ln 3}{\ln 2} \;\Leftrightarrow\; x = 5 + \dfrac{\ln 3}{\ln 2}$

(b) $\ln x + \ln(x - 1) = \ln(x(x - 1)) = 1 \;\Leftrightarrow\; x(x - 1) = e^1 \;\Leftrightarrow\; x^2 - x - e = 0$. The quadratic formula (with $a = 1$, $b = -1$, and $c = -e$) gives $x = \frac{1}{2}\left(1 \pm \sqrt{1 + 4e}\right)$, but we reject the negative root since the natural logarithm is not defined for $x < 0$. So $x = \frac{1}{2}\left(1 + \sqrt{1 + 4e}\right)$.

52. (a) $\ln(\ln x) = 1 \Leftrightarrow e^{\ln(\ln x)} = e^1 \Leftrightarrow \ln x = e^1 = e \Leftrightarrow e^{\ln x} = e^e \Leftrightarrow x = e^e$

(b) $e^{ax} = Ce^{bx} \Leftrightarrow \ln e^{ax} = \ln[C(e^{bx})] \Leftrightarrow ax = \ln C + bx + \ln e^{bx} \Leftrightarrow ax = \ln C + bx \Leftrightarrow$ $ax - bx = \ln C \Leftrightarrow (a-b)x = \ln C \Leftrightarrow x = \dfrac{\ln C}{a-b}$

53. (a) $e^x < 10 \Rightarrow \ln e^x < \ln 10 \Rightarrow x < \ln 10 \Rightarrow x \in (-\infty, \ln 10)$

(b) $\ln x > -1 \Rightarrow e^{\ln x} > e^{-1} \Rightarrow x > e^{-1} \Rightarrow x \in (1/e, \infty)$

54. (a) $2 < \ln x < 9 \Rightarrow e^2 < e^{\ln x} < e^9 \Rightarrow e^2 < x < e^9 \Rightarrow x \in (e^2, e^9)$

(b) $e^{2-3x} > 4 \Rightarrow \ln e^{2-3x} > \ln 4 \Rightarrow 2 - 3x > \ln 4 \Rightarrow -3x > \ln 4 - 2 \Rightarrow$ $x < -\frac{1}{3}(\ln 4 - 2) \Rightarrow x \in \left(-\infty, \frac{1}{3}(2 - \ln 4)\right)$

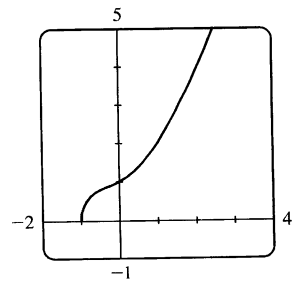
55. (a) For $f(x) = \sqrt{3 - e^{2x}}$, we must have $3 - e^{2x} \geq 0 \Rightarrow e^{2x} \leq 3 \Rightarrow 2x \leq \ln 3 \Rightarrow x \leq \frac{1}{2}\ln 3$. Thus, the domain of f is $(-\infty, \frac{1}{2}\ln 3]$.

(b) $y = f(x) = \sqrt{3 - e^{2x}}$ [note that $y \geq 0$] $\Rightarrow y^2 = 3 - e^{2x} \Rightarrow e^{2x} = 3 - y^2 \Rightarrow 2x = \ln(3 - y^2)$ $\Rightarrow x = \frac{1}{2}\ln(3 - y^2)$. Interchange x and y: $y = \frac{1}{2}\ln(3 - x^2)$. So $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$. For the domain of f^{-1}, we must have $3 - x^2 > 0 \Rightarrow x^2 < 3 \Rightarrow |x| < \sqrt{3} \Rightarrow -\sqrt{3} < x < \sqrt{3} \Rightarrow 0 \leq x < \sqrt{3}$ since $x \geq 0$. Note that the domain of f^{-1}, $[0, \sqrt{3})$, equals the range of f.

56. (a) For $f(x) = \ln(2 + \ln x)$, we must have $2 + \ln x > 0 \Rightarrow \ln x > -2 \Rightarrow x > e^{-2}$. Thus, the domain of f is (e^{-2}, ∞).

(b) $y = f(x) = \ln(2 + \ln x) \Rightarrow e^y = 2 + \ln x \Rightarrow \ln x = e^y - 2 \Rightarrow x = e^{e^y - 2}$. Interchange x and y: $y = e^{e^x - 2}$. So $f^{-1}(x) = e^{e^x - 2}$. The domain of f^{-1}, as well as the range of f, is $\mathbb{R}$.

57. We see that the graph of $y = f(x) = \sqrt{x^3 + x^2 + x + 1}$ is increasing, so f is 1-1. Enter $x = \sqrt{y^3 + y^2 + y + 1}$ and use your CAS to solve the equation for y. Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:



$$y = f^{-1}(x) = -\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$$

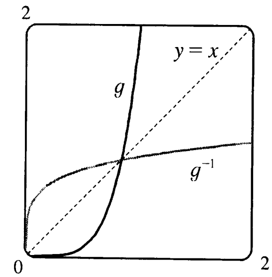
where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$. Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\dfrac{1}{6}\dfrac{M^{2/3} - 8 - 2M^{1/3}}{2M^{1/3}}$, where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.

58. (a) If we use Derive, then solving $x = y^6 + y^4$ for y gives us six solutions of the form $y = \pm\frac{\sqrt{3}}{3}\sqrt{B - 1}$, where $B \in \left\{-2\sin\dfrac{A}{3}, 2\sin\left(\dfrac{A}{3} + \dfrac{\pi}{3}\right), -2\cos\left(\dfrac{A}{3} + \dfrac{\pi}{6}\right)\right\}$ and $A = \sin^{-1}\left(\dfrac{27x - 2}{2}\right)$. The inverse for $y = x^6 + x^4$ ($x \geq 0$) is $y = \frac{\sqrt{3}}{3}\sqrt{B - 1}$ with $B = 2\sin\left(\dfrac{A}{3} + \dfrac{\pi}{3}\right)$, but because the domain of A is $\left[0, \frac{4}{27}\right]$, this expression is only valid for $x \in \left[0, \frac{4}{27}\right]$.

Happily, Maple gives us the rest of the solution! We solve $x = y^6 + y^4$ for y to get the two real solutions $\pm\frac{\sqrt{6}}{6}\frac{\sqrt{C^{1/3}\left(C^{2/3} - 2C^{1/3} + 4\right)}}{C^{1/3}}$, where $C = 108x + 12\sqrt{3}\sqrt{x\,(27x - 4)}$, and the inverse for $y = x^6 + x^4$ ($x \geq 0$) is the positive solution, whose domain is $\left[\frac{4}{27}, \infty\right)$.

Mathematica also gives two real solutions, equivalent to those of Maple. The positive one is $\frac{\sqrt{6}}{6}\left(\sqrt[3]{4}D^{1/3} + 2\sqrt[3]{2}D^{-1/3} - 2\right)$, where $D = -2 + 27x + 3\sqrt{3}\sqrt{x}\sqrt{27x - 4}$. Although this expression also has domain $\left[\frac{4}{27}, \infty\right)$, Mathematica is mysteriously able to plot the solution for all $x \geq 0$.

(b)



59. (a) $n = 100 \cdot 2^{t/3} \;\Rightarrow\; \frac{n}{100} = 2^{t/3} \;\Rightarrow\; \log_2\left(\frac{n}{100}\right) = \frac{t}{3} \;\Rightarrow\; t = 3\log_2\left(\frac{n}{100}\right)$. Using formula (10), we can write this as $t = 3 \cdot \frac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n = 50{,}000 \;\Rightarrow\; t = 3\log_2 \frac{50{,}000}{100} = 3\log_2 500 = 3\left(\frac{\ln 500}{\ln 2}\right) \approx 26.9$ hours

60. (a) $Q = Q_0\left(1 - e^{-t/a}\right) \;\Rightarrow\; \frac{Q}{Q_0} = 1 - e^{-t/a} \;\Rightarrow\; e^{-t/a} = 1 - \frac{Q}{Q_0} \;\Rightarrow\; -\frac{t}{a} = \ln\left(1 - \frac{Q}{Q_0}\right) \;\Rightarrow$
$t = -a\ln(1 - Q/Q_0)$. This gives us the time t necessary to obtain a given charge Q.

(b) $Q = 0.9Q_0$ and $a = 2 \;\Rightarrow\; t = -2\ln\left(1 - 0.9(Q_0/Q_0)\right) = -2\ln 0.1 \approx 4.6$ seconds.

61. (a) To find the equation of the graph that results from shifting the graph of $y = \ln x$ 3 units upward, we add 3 to the original function to get $y = \ln x + 3$.

(b) To find the equation of the graph that results from shifting the graph of $y = \ln x$ 3 units to the left, we replace x with $x + 3$ in the original function to get $y = \ln(x + 3)$.

(c) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the x-axis, we multiply the original equation by -1 to get $y = -\ln x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the y-axis, we replace x with $-x$ in the original equation to get $y = \ln(-x)$.

(e) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the line $y = x$, we interchange x and y in the original equation to get $x = \ln y \;\Leftrightarrow\; y = e^x$.

(f) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the x-axis and then about the line $y = x$, we first multiply the original equation by -1 [to get $y = -\ln x$] and then interchange x and y in this equation to get $x = -\ln y \;\Leftrightarrow\; \ln y = -x \;\Leftrightarrow\; y = e^{-x}$.

(g) To find the equation of the graph that results from reflecting the graph of $y = \ln x$ about the y-axis and then about the line $y = x$, we first replace x with $-x$ in the original equation [to get $y = \ln(-x)$] and then interchange x and y to get $x = \ln(-y) \;\Leftrightarrow\; -y = e^x \;\Leftrightarrow\; y = -e^x$.

(h) To find the equation of the graph that results from shifting the graph of $y = \ln x$ 3 units to the left and then reflecting it about the line $y = x$, we first replace x with $x + 3$ in the original equation [to get $y = \ln(x + 3)$] and then interchange x and y in this equation to get $x = \ln(y + 3) \;\Leftrightarrow\; y + 3 = e^x \;\Leftrightarrow\; y = e^x - 3$.

62. (a) If the point (x, y) is on the graph of $y = f(x)$, then the point $(x - c, y)$ is that point shifted c units to the left. Since f is 1-1, the point (y, x) is on the graph of $y = f^{-1}(x)$ and the point corresponding to $(x - c, y)$ on the graph of f is $(y, x - c)$ on the graph of f^{-1}. Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x) = f^{-1}(x) - c$.

(b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y = x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x) = f(cx)$ can be expressed as $h^{-1}(x) = (1/c)\, f^{-1}(x)$.

63. (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\cos^{-1}(-1) = \pi$ since $\cos\pi = -1$ and π is in $[0, \pi]$.

64. (a) $\arctan(-1) = -\frac{\pi}{4}$ since $\tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\csc^{-1} 2 = \frac{\pi}{6}$ since $\csc\frac{\pi}{6} = 2$ and $\frac{\pi}{6}$ is in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.

65. (a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ since $\tan\frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

66. (a) $\sec^{-1}\sqrt{2} = \frac{\pi}{4}$ since $\sec\frac{\pi}{4} = \sqrt{2}$ and $\frac{\pi}{4}$ is in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.

(b) $\arcsin 1 = \frac{\pi}{2}$ since $\sin\frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

67. (a) $\sin(\sin^{-1} 0.7) = 0.7$ since 0.7 is in $[-1, 1]$.

(b) $\tan^{-1}\left(\tan\frac{4\pi}{3}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$ since $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

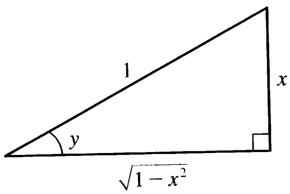
68. (a) Let $\theta = \arctan 2$, so $\tan\theta = 2 \Rightarrow \sec^2\theta = 1 + \tan^2\theta = 1 + 4 = 5 \Rightarrow \sec\theta = \sqrt{5} \Rightarrow$ $\sec(\arctan\ 2) = \sec\theta = \sqrt{5}$.

(b) Let $\theta = \sin^{-1}\frac{5}{13}$. Then $\sin\theta = \frac{5}{13}$, so $\cos\left(2\sin^{-1}\frac{5}{13}\right) = \cos 2\theta = 1 - 2\sin^2\theta = 1 - 2\left(\frac{5}{13}\right)^2 = \frac{119}{169}$.

69. Let $y = \sin^{-1} x$. Then $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y \ge 0$, so $\cos(\sin^{-1} x) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

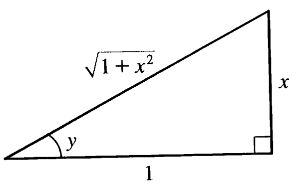
70. Let $y = \sin^{-1} x$. Then $\sin y = x$, so from the triangle we see that

$$\tan(\sin^{-1} x) = \tan y = \frac{x}{\sqrt{1 - x^2}}.$$



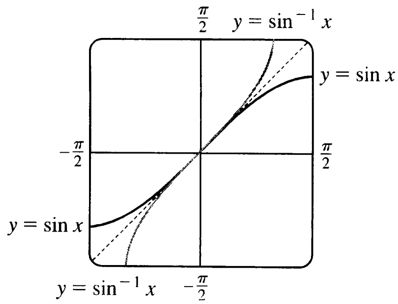
71. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that

$$\sin(\tan^{-1} x) = \sin y = \frac{x}{\sqrt{1 + x^2}}.$$



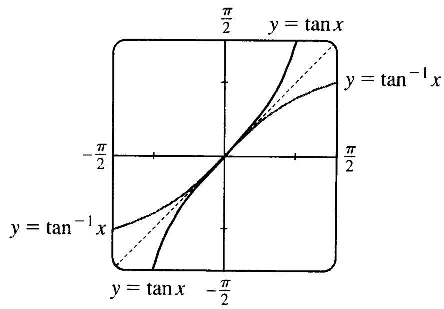
72. Let $y = \cos^{-1} x$. Then $\cos y = x \Rightarrow \sin y = \sqrt{1 - x^2}$ since $0 \le y \le \pi$. So $\sin(2\cos^{-1} x) = \sin 2y = 2\sin y\cos y = 2x\sqrt{1 - x^2}$.

73.



The graph of $\sin^{-1} x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

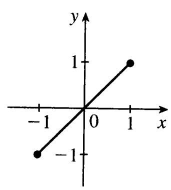
74.



The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y = x$.

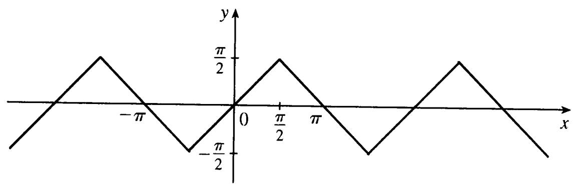
75. $g(x) = \sin^{-1}(3x + 1)$.
Domain $(g) = \{x \mid -1 \leq 3x + 1 \leq 1\} = \{x \mid -2 \leq 3x \leq 0\} = \{x \mid -\frac{2}{3} \leq x \leq 0\} = [-\frac{2}{3}, 0]$.
Range $(g) = \{y \mid -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\} = [-\frac{\pi}{2}, \frac{\pi}{2}]$.

76. (a) $f(x) = \sin(\sin^{-1} x)$



Since one function undoes what the other one does, we get the identity function, $y = x$, on the restricted domain $-1 \leq x \leq 1$.

(b) $g(x) = \sin^{-1}(\sin x)$



This is similar to part (a), but with domain $\mathbb{R}$. Equations for g on intervals of the form $\left(-\frac{\pi}{2} + \pi n, \frac{\pi}{2} + \pi n\right)$, for any integer n, can be found using $g(x) = (-1)^n x + (-1)^{n+1} n\pi$. The sine function is monotonic on each of these intervals, and hence, so is g (but in a linear fashion).

1 Review

CONCEPT CHECK

1. (a) A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B. The set A is called the **domain** of the function. The **range** of f is the set of all possible values of $f(x)$ as x varies throughout the domain.

(b) If f is a function with domain A, then its **graph** is the set of ordered pairs $\{(x, f(x)) \mid x \in A\}$.

(c) Use the Vertical Line Test on page 17.

2. The four ways to represent a function are: verbally, numerically, visually, and algebraically. An example of each is given below.

Verbally: An assignment of students to chairs in a classroom (a description in words)

Numerically: A tax table that assigns an amount of tax to an income (a table of values)

Visually: A graphical history of the Dow Jones average (a graph)

Algebraically: A relationship between distance, rate, and time: $d = rt$ (an explicit formula)

3. (a) An **even function** f satisfies $f(-x) = f(x)$ for every number x in its domain. It is symmetric with respect to the y-axis.

   (b) An **odd function** g satisfies $g(-x) = -g(x)$ for every number x in its domain. It is symmetric with respect to the origin.

4. A function f is called **increasing** on an interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

5. A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real-world phenomenon.

6. (a) Linear function: $f(x) = 2x + 1$, $f(x) = ax + b$

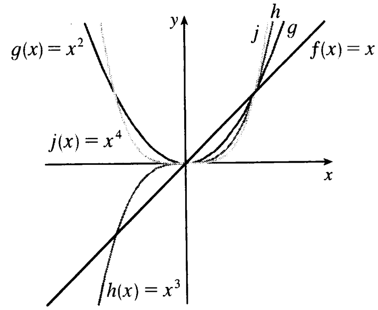
   (b) Power function: $f(x) = x^2$, $f(x) = x^a$

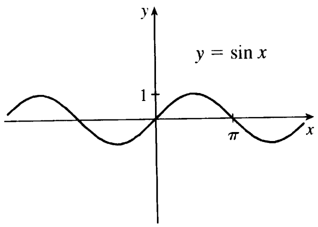
   (c) Exponential function: $f(x) = 2^x$, $f(x) = a^x$

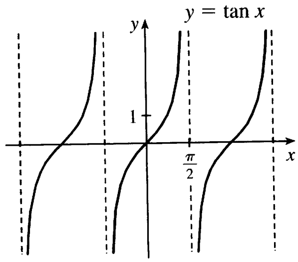
   (d) Quadratic function: $f(x) = x^2 + x + 1$, $f(x) = ax^2 + bx + c$

   (e) Polynomial of degree 5: $f(x) = x^5 + 2$

   (f) Rational function: $f(x) = \dfrac{x}{x+2}$, $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials

7.


8. (a)


(b)


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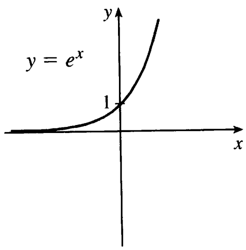

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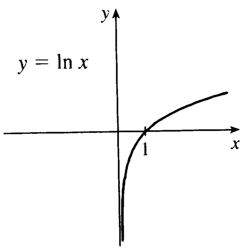
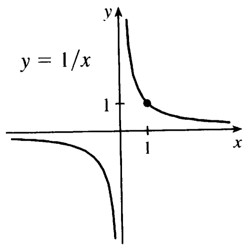
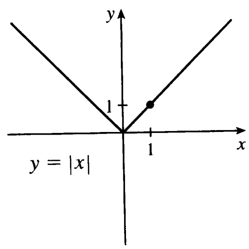
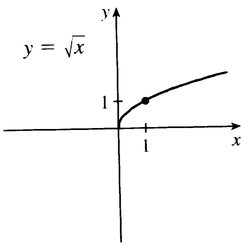
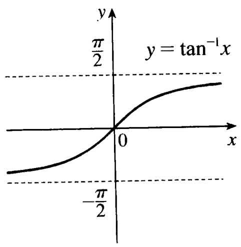


(e)


(f)


(g)


(h)


9. (a) The domain of $f + g$ is the intersection of the domain of f and the domain of g; that is, $A \cap B$.

(b) The domain of fg is also $A \cap B$.

(c) The domain of f/g must exclude values of x that make g equal to 0; that is, $\{x \in A \cap B \mid g(x) \neq 0\}$.

10. Given two functions f and g, the **composite** function $f \circ g$ is defined by $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f.

11. (a) If the graph of f is shifted 2 units upward, its equation becomes $y = f(x) + 2$.

(b) If the graph of f is shifted 2 units downward, its equation becomes $y = f(x) - 2$.

(c) If the graph of f is shifted 2 units to the right, its equation becomes $y = f(x - 2)$.

(d) If the graph of f is shifted 2 units to the left, its equation becomes $y = f(x + 2)$.

(e) If the graph of f is reflected about the x-axis, its equation becomes $y = -f(x)$.

(f) If the graph of f is reflected about the y-axis, its equation becomes $y = f(-x)$.

(g) If the graph of f is stretched vertically by a factor of 2, its equation becomes $y = 2f(x)$.

(h) If the graph of f is shrunk vertically by a factor of 2, its equation becomes $y = \frac{1}{2}f(x)$.

(i) If the graph of f is stretched horizontally by a factor of 2, its equation becomes $y = f(\frac{1}{2})x$.

(j) If the graph of f is shrunk horizontally by a factor of 2, its equation becomes $y = f(2x)$.

12. (a) A function f is called a *one-to-one function* if it never takes on the same value twice; that is, if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$. (Or, f is 1-1 if each output corresponds to only one input.)
Use the Horizontal Line Test: A function is one-to-one if and only if no horizontal line intersects its graph more than once.

(b) If f is a one-to-one function with domain A and range B, then its *inverse function* f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

13. (a) The inverse sine function $f(x) = \sin^{-1} x$ is defined as follows:

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x \quad \textit{and} \quad -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

Its domain is $-1 \leq x \leq 1$ and its range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

(b) The inverse cosine function $f(x) = \cos^{-1} x$ is defined as follows:

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x \quad \textit{and} \quad 0 \leq y \leq \pi$$

Its domain is $-1 \leq x \leq 1$ and its range is $0 \leq y \leq \pi$.

(c) The inverse tangent function $f(x) = \tan^{-1} x$ is defined as follows:

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x \quad \textit{and} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

Its domain is $\mathbb{R}$ and its range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

TRUE-FALSE QUIZ

1. False. Let $f(x) = x^2$, $s = -1$, and $t = 1$. Then $f(s+t) = (-1+1)^2 = 0^2 = 0$, but $f(s) + f(t) = (-1)^2 + 1^2 = 2 \neq 0 = f(s+t)$.

2. False. Let $f(x) = x^2$. Then $f(-2) = 4 = f(2)$, but $-2 \neq 2$.

3. False. Let $f(x) = x^2$. Then $f(3x) = (3x)^2 = 9x^2$ and $3f(x) = 3x^2$. So $f(3x) \neq 3f(x)$.

4. True. If $x_1 < x_2$ and f is a decreasing function, then the y-values get smaller as we move from left to right. Thus, $f(x_1) > f(x_2)$.

5. True. See the Vertical Line Test.

6. False. Let $f(x) = x^2$ and $g(x) = 2x$. Then $(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 = 4x^2$ and $(g \circ f)(x) = g(f(x)) = g(x^2) = 2x^2$. So $f \circ g \neq g \circ f$.

7. False. Let $f(x) = x^3$. Then f is one-to-one and $f^{-1}(x) = \sqrt[3]{x}$. But $1/f(x) = 1/x^3$, which is not equal to $f^{-1}(x)$.

8. True. We can divide by e^x since $e^x \neq 0$ for every x.

9. True. The function $\ln x$ is an increasing function on $(0, \infty)$.

10. False. Let $x = e$. Then $(\ln x)^6 = (\ln e)^6 = 1^6 = 1$, but $6 \ln x = 6 \ln e = 6 \cdot 1 = 6 \neq 1 = (\ln x)^6$.

11. False. Let $x = e^2$ and $a = e$. Then $\dfrac{\ln x}{\ln a} = \dfrac{\ln e^2}{\ln e} = \dfrac{2 \ln e}{\ln e} = 2$ and $\ln \dfrac{x}{a} = \ln \dfrac{e^2}{e} = \ln e = 1$, so in general the statement is false. What *is* true, however, is that $\ln \dfrac{x}{a} = \ln x - \ln a$.

EXERCISES

1. (a) When $x = 2$, $y \approx 2.7$. Thus, $f(2) \approx 2.7$.

(b) $f(x) = 3 \quad \Rightarrow \quad x \approx 2.3, 5.6$

(c) The domain of f is $-6 \leq x \leq 6$, or $[-6, 6]$.

(d) The range of f is $-4 \leq y \leq 4$, or $[-4, 4]$.

(e) f is increasing on $[-4, 4]$, that is, on $-4 \leq x \leq 4$.

(f) f is not one-to-one since it fails the Horizontal Line Test.

(g) f is odd since its graph is symmetric about the origin.

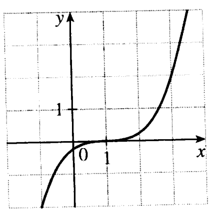
2. (a) When $x = 2$, $y = 3$. Thus, $g(2) = 3$.

(b) g is one-to-one because it passes the Horizontal Line Test.

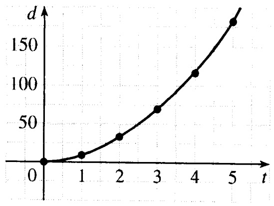
(c) When $y = 2$, $x \approx 0.2$. So $g^{-1}(2) \approx 0.2$.

(d) The range of g is $[-1, 3.5]$, which is the same as the domain of g^{-1}.

(e) We reflect the graph of g through the line $y = x$ to obtain the graph of g^{-1}.

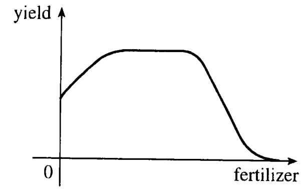


3. (a)



(b) From the graph, we see that the distance traveled after 4.5 seconds is slightly less than 150 feet.

4.



There will be some yield with no fertilizer, increasing yields with increasing fertilizer use, a leveling-off of yields at some point, and disaster with too much fertilizer use.

5. $f(x) = \sqrt{4 - 3x^2}$. Domain: $4 - 3x^2 \geq 0 \quad \Rightarrow \quad 3x^2 \leq 4 \quad \Rightarrow \quad x^2 \leq \frac{4}{3} \quad \Rightarrow \quad |x| \leq \frac{2}{\sqrt{3}}$. Range: $y \geq 0$ and $y \leq \sqrt{4} \quad \Rightarrow \quad 0 \leq y \leq 2$.

6. $g(x) = \dfrac{1}{x+1}$. Domain: $x + 1 \neq 0 \quad \Rightarrow \quad x \neq -1$. Range: all reals except 0 ($y = 0$ is the horizontal asymptote for g.)

7. $y = 1 + \sin x$. Domain: $\mathbb{R}$. Range: $-1 \leq \sin x \leq 1 \quad \Rightarrow \quad 0 \leq 1 + \sin x \leq 2 \quad \Rightarrow \quad 0 \leq y \leq 2$.

8. $y = \ln \ln x$. Domain: We must have $\ln x > 0 \quad \Rightarrow \quad x > e^0 \quad \Rightarrow \quad x > 1$. Range: $\ln x > 0$, so $\ln(\ln x)$ takes on all real numbers and, hence, the range is $\mathbb{R}$.

9. (a) To obtain the graph of $y = f(x) + 8$, we shift the graph of $y = f(x)$ up 8 units.

(b) To obtain the graph of $y = f(x + 8)$, we shift the graph of $y = f(x)$ left 8 units.

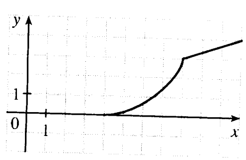
(c) To obtain the graph of $y = 1 + 2f(x)$, we stretch the graph of $y = f(x)$ vertically by a factor of 2, and then shift the resulting graph 1 unit upward.

(d) To obtain the graph of $y = f(x - 2) - 2$, we shift the graph of $y = f(x)$ right 2 units (for the "-2" inside the parentheses), and then shift the resulting graph 2 units downward.

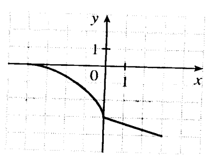
(e) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.

(f) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$ (assuming f is one–to-one).

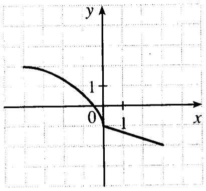
10. (a) To obtain the graph of $y = f(x - 8)$, we shift the graph of $y = f(x)$ right 8 units.



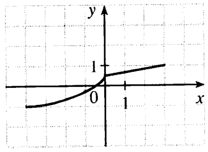
(b) To obtain the graph of $y = -f(x)$, we reflect the graph of $y = f(x)$ about the x-axis.



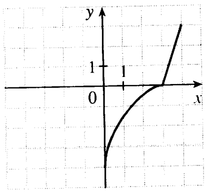
(c) To obtain the graph of $y = 2 - f(x)$, we reflect the graph of $y = f(x)$ about the x-axis, and then shift the resulting graph 2 units upward.



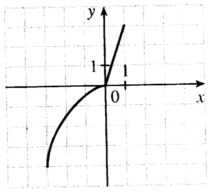
(d) To obtain the graph of $y = \frac{1}{2}f(x) - 1$, we shrink the graph of $y = f(x)$ by a factor of 2, and then shift the resulting graph 1 unit downward.



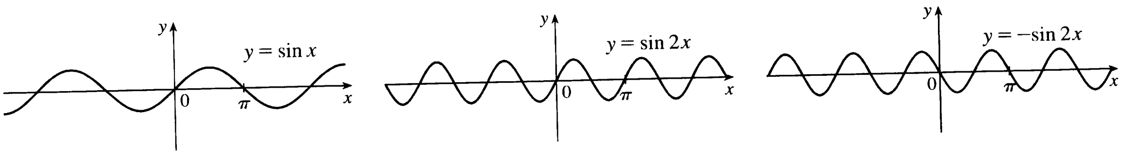
(e) To obtain the graph of $y = f^{-1}(x)$, we reflect the graph of $y = f(x)$ about the line $y = x$.



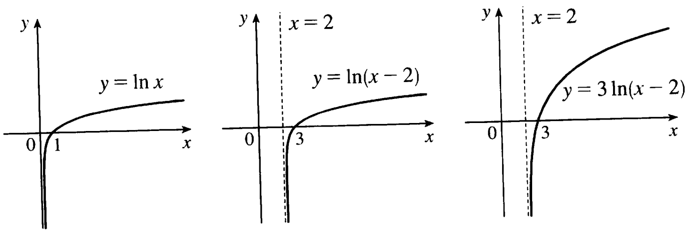
(f) To obtain the graph of $y = f^{-1}(x+3)$, we reflect the graph of $y = f(x)$ about the line $y = x$ [see part (e)], and then shift the resulting graph left 3 units.



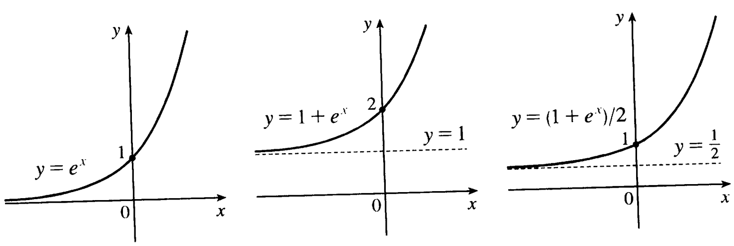
11. $y = -\sin 2x$: Start with the graph of $y = \sin x$, compress horizontally by a factor of 2, and reflect about the x-axis.



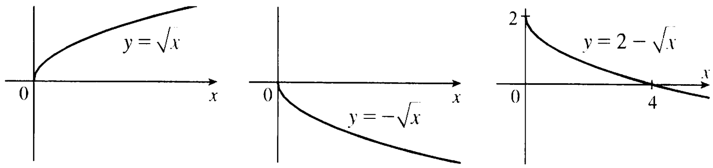
12. $y = 3\ln(x-2)$: Start with the graph of $y = \ln x$, shift 2 units to the right, and stretch vertically by a factor of 3.



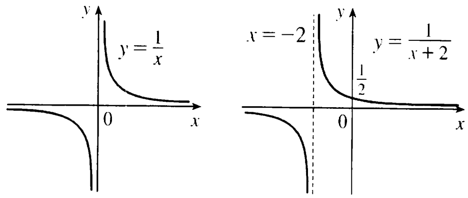
13. $y = (1 + e^x)/2$: Start with the graph of $y = e^x$, shift 1 unit upward, and compress vertically by a factor of 2.



14. $y = 2 - \sqrt{x}$: Start with the graph of $y = \sqrt{x}$, reflect about the x-axis, and shift 2 units upward.



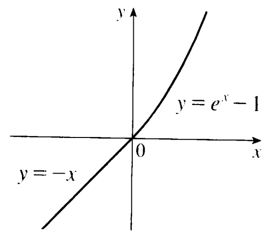
15. $f(x) = \dfrac{1}{x+2}$: Start with the graph of $f(x) = 1/x$ and shift 2 units to the left.



16. $f(x) = \begin{cases} -x & \text{if } x < 0 \\ e^x - 1 & \text{if } x \geq 0 \end{cases}$

On $(-\infty, 0)$, graph $y = -x$ (the line with slope -1 and y-intercept 0) with open endpoint $(0, 0)$.

On $[0, \infty)$, graph $y = e^x - 1$ (the graph of $y = e^x$ shifted 1 unit downward) with closed endpoint $(0, 0)$.



17. (a) The terms of f are a mixture of odd and even powers of x, so f is neither even nor odd.

(b) The terms of f are all odd powers of x, so f is odd.

(c) $f(-x) = e^{-(-x)^2} = e^{-x^2} = f(x)$, so f is even.

(d) $f(-x) = 1 + \sin(-x) = 1 - \sin x$. Now $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so f is neither even nor odd.

18. For the line segment from $(-2, 2)$ to $(-1, 0)$, the slope is $\dfrac{0-2}{-1+2} = -2$, and an equation is $y - 0 = -2\,(x+1)$ or, equivalently, $y = -2x - 2$. The circle has equation $x^2 + y^2 = 1$; the top half has equation $y = \sqrt{1 - x^2}$ (we have solved for positive y.) Thus, $f(x) = \begin{cases} -2x - 2 & \text{if } -2 \leq x \leq -1 \\ \sqrt{1 - x^2} & \text{if } -1 < x \leq 1 \end{cases}$

19. $f(x) = \ln x$, $D = (0, \infty)$; $g(x) = x^2 - 9$, $D = \mathbb{R}$.

$(f \circ g)(x) = f(g(x)) = f(x^2 - 9) = \ln(x^2 - 9)$.

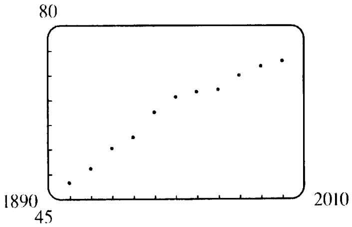
Domain: $x^2 - 9 > 0 \Rightarrow x^2 > 9 \Rightarrow |x| > 3 \Rightarrow x \in (-\infty, -3) \cup (3, \infty)$

$(g \circ f)(x) = g(f(x)) = g(\ln x) = (\ln x)^2 - 9$. Domain: $x > 0$, or $(0, \infty)$

$(f \circ f)(x) = f(f(x)) = f(\ln x) = \ln(\ln x)$. Domain: $\ln x > 0 \Rightarrow x > e^0 = 1$, or $(1, \infty)$

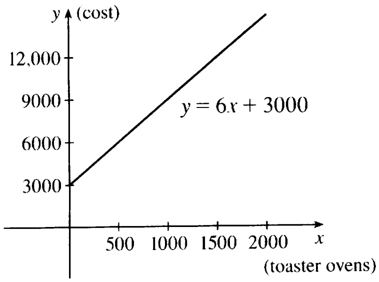
$(g \circ g)(x) = g(g(x)) = g(x^2 - 9) = (x^2 - 9)^2 - 9$. Domain: $x \in \mathbb{R}$, or $(-\infty, \infty)$

20. Let $h(x) = x + \sqrt{x}$, $g(x) = \sqrt{x}$, and $f(x) = 1/x$. Then $(f \circ g \circ h)(x) = \dfrac{1}{\sqrt{x + \sqrt{x}}} = F(x)$.

21. 

Many models appear to be plausible. Your choice depends on whether you think medical advances will keep increasing life expectancy, or if there is bound to be a natural leveling-off of life expectancy. A linear model, $y = 0.2493x - 423.4818$ gives us an estimate of 77.6 years for the year 2010.

22. (a) Let x denote the number of toaster ovens produced in one week and y the associated cost. Using the points $(1000, 9000)$ and $(1500, 12{,}000)$, we get an equation of a line: $y - 9000 = \dfrac{12{,}000 - 9000}{1500 - 1000}(x - 1000) \Rightarrow$ $y = 6(x - 1000) + 9000 \Rightarrow y = 6x + 3000$.



(b) The slope of 6 means that each additional toaster oven produced adds \$6 to the weekly production cost.

(c) The y-intercept of 3000 represents the overhead cost—the cost incurred without producing anything.

23. We need to know the value of x such that $f(x) = 2x + \ln x = 2$. Since $x = 1$ gives us $y = 2$, $f^{-1}(2) = 1$.

24. $y = \dfrac{x+1}{2x+1}$. Interchanging x and y gives us $x = \dfrac{y+1}{2y+1} \Rightarrow 2xy + x = y + 1 \Rightarrow 2xy - y = 1 - x \Rightarrow$ $y(2x - 1) = 1 - x \Rightarrow y = \dfrac{1-x}{2x-1} = f^{-1}(x)$.

25. (a) $e^{2\ln 3} = \left(e^{\ln 3}\right)^2 = 3^2 = 9$

(b) $\log_{10} 25 + \log_{10} 4 = \log_{10}(25 \cdot 4) = \log_{10} 100 = \log_{10} 10^2 = 2$

(c) $\tan\left(\arcsin \frac{1}{2}\right) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

(d) Let $\theta = \cos^{-1} \frac{4}{5}$, so $\cos\theta = \frac{4}{5}$. Then $\sin\left(\cos^{-1} \frac{4}{5}\right) = \sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$.

26. (a) $e^x = 5 \Rightarrow x = \ln 5$

(b) $\ln x = 2 \Rightarrow x = e^2$

(c) $e^{e^x} = 2 \Rightarrow e^x = \ln 2 \Rightarrow x = \ln(\ln 2)$

(d) $\tan^{-1} x = 1 \Rightarrow \tan\tan^{-1} x = \tan 1 \Rightarrow x = \tan 1\ (\approx 1.5574)$

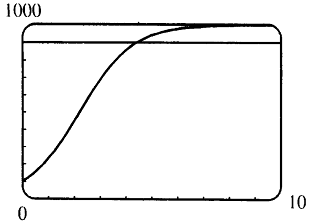
27. (a) After 4 days, $\frac{1}{2}$ gram remains; after 8 days, $\frac{1}{4}$ g; after 12 days, $\frac{1}{8}$ g; after 16 days, $\frac{1}{16}$ g.

(b) $m(4) = \dfrac{1}{2}$, $m(8) = \dfrac{1}{2^2}$, $m(12) = \dfrac{1}{2^3}$, $m(16) = \dfrac{1}{2^4}$. From the pattern, we see that $m(t) = \dfrac{1}{2^{t/4}}$, or $2^{-t/4}$.

(c) $m = 2^{-t/4} \Rightarrow \log_2 m = -t/4 \Rightarrow t = -4\log_2 m$; this is the time elapsed when there are m grams of ^{100}Pd.

(d) $m = 0.01 \Rightarrow t = -4\log_2 0.01 = -4\left(\dfrac{\ln 0.01}{\ln 2}\right) \approx 26.6$ days

28. (a)



The population would reach 900 in about 4.4 years.

(b) $P = \dfrac{100{,}000}{100 + 900e^{-t}} \Rightarrow \quad 100P + 900Pe^{-t} = 100{,}000 \Rightarrow$

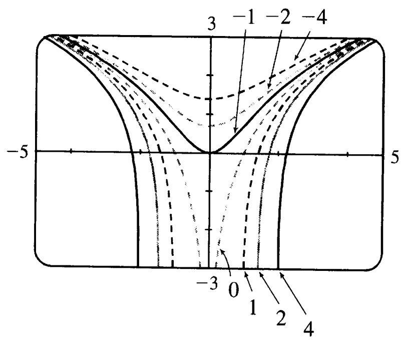
$900Pe^{-t} = 100{,}000 - 100P \quad \Rightarrow \quad e^{-t} = \dfrac{100{,}000 - 100P}{900P} \Rightarrow$

$-t = \ln\left(\dfrac{1000 - P}{9P}\right) \quad \Rightarrow \quad t = -\ln\left(\dfrac{1000 - P}{9P}\right)$, or

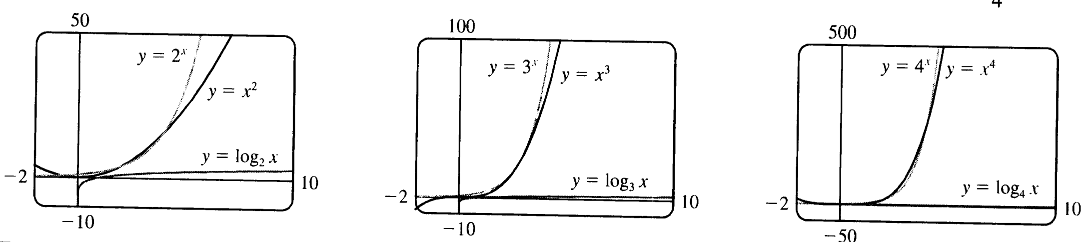
$\ln\left(\dfrac{9P}{1000 - P}\right)$; this is the time required for the population to reach a given number P.

(c) $P = 900 \quad \Rightarrow \quad t = \ln\left(\dfrac{9 \cdot 900}{1000 - 900}\right) = \ln 81 \approx 4.4$ years, as in part (a).

29. $f(x) = \ln(x^2 - c)$. If $c < 0$, the domain of f is $\mathbb{R}$. If $c = 0$, the domain of f is $(-\infty, 0) \cup (0, \infty)$. If $c > 0$, the domain of f is $(-\infty, -\sqrt{c}) \cup (\sqrt{c}, \infty)$. As c increases, the dip at $x = 0$ becomes deeper. For $c \geq 0$, the graph has asymptotes at $x = \pm\sqrt{c}$.



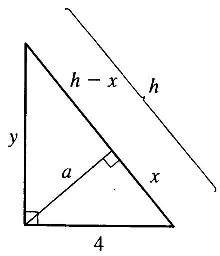
30.



For large values of x, $y = a^x$ has the largest y-values and $y = \log_a x$ has the smallest y-values. This makes sense because they are inverses of each other.

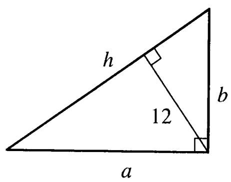
PRINCIPLES OF PROBLEM SOLVING

1.



By using the area formula for a triangle, $\frac{1}{2}$ (base) (height), in two ways, we see that $\frac{1}{2}(4)(y) = \frac{1}{2}(h)(a)$, so $a = \dfrac{4y}{h}$. Since $4^2 + y^2 = h^2$, $y = \sqrt{h^2 - 16}$, and $a = \dfrac{4\sqrt{h^2 - 16}}{h}$.

2.



Refer to Example 1, where we obtained $h = \dfrac{P^2 - 100}{2P}$. The 100 came from 4 times the area of the triangle.

In this case, the area of the triangle is $\frac{1}{2}(h)(12) = 6h$. Thus, $h = \dfrac{P^2 - 4(6h)}{2P} \;\Rightarrow\; 2Ph = P^2 - 24h \;\Rightarrow$ $2Ph + 24h = P^2 \;\Rightarrow\; h(2P + 24) = P^2 \;\Rightarrow\; h = \dfrac{P^2}{2P + 24}$.

3. $|2x - 1| = \begin{cases} 2x - 1 & \text{if } x \geq \frac{1}{2} \\ 1 - 2x & \text{if } x < \frac{1}{2} \end{cases}$ and $|x + 5| = \begin{cases} x + 5 & \text{if } x \geq -5 \\ -x - 5 & \text{if } x < -5 \end{cases}$

Therefore, we consider the three cases $x < -5$, $-5 \leq x < \frac{1}{2}$, and $x \geq \frac{1}{2}$.

If $x < -5$, we must have $1 - 2x - (-x - 5) = 3 \;\Leftrightarrow\; x = 3$, which is false, since we are considering $x < -5$.

If $-5 \leq x < \frac{1}{2}$, we must have $1 - 2x - (x + 5) = 3 \;\Leftrightarrow\; x = -\frac{7}{3}$.

If $x \geq \frac{1}{2}$, we must have $2x - 1 - (x + 5) = 3 \;\Leftrightarrow\; x = 9$.

So the two solutions of the equation are $x = -\frac{7}{3}$ and $x = 9$.

4. $|x - 1| = \begin{cases} x - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } x < 1 \end{cases}$ and $|x - 3| = \begin{cases} x - 3 & \text{if } x \geq 3 \\ 3 - x & \text{if } x < 3 \end{cases}$

Therefore, we consider the three cases $x < 1$, $1 \leq x < 3$, and $x \geq 3$.

If $x < 1$, we must have $1 - x - (3 - x) \geq 5 \;\Leftrightarrow\; 0 \geq 7$, which is false.

If $1 \leq x < 3$, we must have $x - 1 - (3 - x) \geq 5 \;\Leftrightarrow\; x \geq \frac{9}{2}$, which is false because $x < 3$.

If $x \geq 3$, we must have $x - 1 - (x - 3) \geq 5 \;\Leftrightarrow\; 2 \geq 5$, which is false.

All three cases lead to falsehoods, so the inequality has no solution.

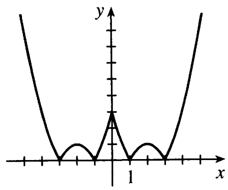
5. $f(x) = \left|x^2 - 4|x| + 3\right|$. If $x \geq 0$, then $f(x) = \left|x^2 - 4x + 3\right| = |(x-1)(x-3)|$.

Case (i): If $0 < x \leq 1$, then $f(x) = x^2 - 4x + 3$.

Case (ii): If $1 < x \leq 3$, then $f(x) = -(x^2 - 4x + 3) = -x^2 + 4x - 3$.

Case (iii): If $x > 3$, then $f(x) = x^2 - 4x + 3$.

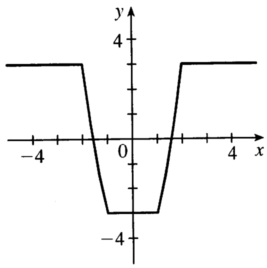
This enables us to sketch the graph for $x \geq 0$. Then we use the fact that f is an even function to reflect this part of the graph about the y-axis to obtain the entire graph. Or, we could consider also the cases $x < -3$, $-3 \leq x < -1$, and $-1 \leq x < 0$.



6. $g(x) = \left|x^2 - 1\right| - \left|x^2 - 4\right|$.

$$\left|x^2 - 1\right| = \begin{cases} x^2 - 1 & \text{if } |x| \geq 1 \\ 1 - x^2 & \text{if } |x| < 1 \end{cases} \quad \text{and} \quad \left|x^2 - 4\right| = \begin{cases} x^2 - 4 & \text{if } |x| \geq 2 \\ 4 - x^2 & \text{if } |x| < 2 \end{cases}$$

So for $0 \leq |x| < 1$, $g(x) = 1 - x^2 - (4 - x^2) = -3$, for $1 \leq |x| < 2$, $g(x) = x^2 - 1 - (4 - x^2) = 2x^2 - 5$, and for $|x| \geq 2$, $g(x) = x^2 - 1 - (x^2 - 4) = 3$.



7. Remember that $|a| = a$ if $a \geq 0$ and that $|a| = -a$ if $a < 0$. Thus,

$$x + |x| = \begin{cases} 2x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \quad \text{and} \quad y + |y| = \begin{cases} 2y & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

We will consider the equation $x + |x| = y + |y|$ in four cases.

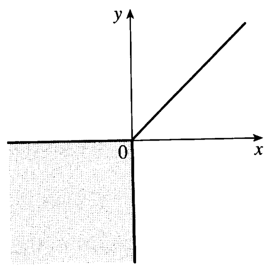
(1) $x \geq 0, y \geq 0$	(2) $x \geq 0, y < 0$	(3) $x < 0, y \geq 0$	(4) $x < 0, y < 0$
$2x = 2y$	$2x = 0$	$0 = 2y$	$0 = 0$
$x = y$	$x = 0$	$0 = y$	

Case 1 gives us the line $y = x$ with nonnegative x and y.

Case 2 gives us the portion of the y-axis with y negative.

Case 3 gives us the portion of the x-axis with x negative.

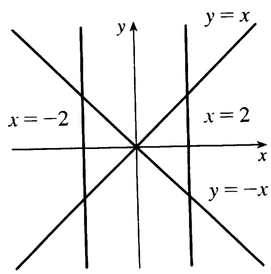
Case 4 gives us the entire third quadrant.



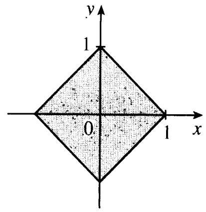
8. $x^4 - 4x^2 - x^2y^2 + 4y^2 = 0 \quad \Leftrightarrow \quad x^2(x^2 - 4) - y^2(x^2 - 4) = 0 \quad \Leftrightarrow$

$(x^2 - y^2)(x^2 - 4) = 0 \quad \Leftrightarrow \quad (x + y)(x - y)(x + 2)(x - 2) = 0$.

So the graph of the equation consists of the graphs of the four lines $y = -x$, $y = x$, $x = -2$, and $x = 2$.



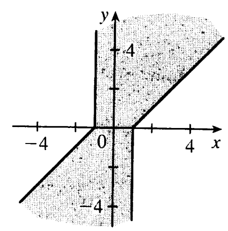
9. $|x| + |y| \leq 1$. The boundary of the region has equation $|x| + |y| = 1$. In quadrants I, II, III, and IV, this becomes the lines $x + y = 1$, $-x + y = 1$, $-x - y = 1$, and $x - y = 1$ respectively.



10. $|x - y| + |x| - |y| \leq 2$

Case (i):	$x > y > 0$	$\Leftrightarrow$	$x - y + x - y \leq 2$	$\Leftrightarrow$	$x - y \leq 1$	$\Leftrightarrow$	$y \geq x - 1$
Case (ii):	$y > x > 0$	$\Leftrightarrow$	$y - x + x - y \leq 2$	$\Leftrightarrow$	$0 \leq 2$ (true)		
Case (iii):	$x > 0$ and $y < 0$	$\Leftrightarrow$	$x - y + x + y \leq 2$	$\Leftrightarrow$	$2x \leq 2$	$\Leftrightarrow$	$x \leq 1$
Case (iv):	$x < 0$ and $y > 0$	$\Leftrightarrow$	$y - x - x - y \leq 2$	$\Leftrightarrow$	$-2x \leq 2$	$\Leftrightarrow$	$x \geq -1$
Case (v):	$y < x < 0$	$\Leftrightarrow$	$x - y - x + y \leq 2$	$\Leftrightarrow$	$0 \leq 2$ (true)		
Case (vi):	$x < y < 0$	$\Leftrightarrow$	$y - x - x + y \leq 2$	$\Leftrightarrow$	$y - x \leq 1$	$\Leftrightarrow$	$y \leq x + 1$

Note: Instead of considering cases (iv), (v), and (vi), we could have noted that the region is unchanged if x and y are replaced by $-x$ and $-y$, so the region is symmetric about the origin. Therefore, we need only draw cases (i), (ii), and (iii), and rotate through $180°$ about the origin.



11. $(\log_2 3)(\log_3 4)(\log_4 5) \cdots (\log_{31} 32) = \left(\dfrac{\ln 3}{\ln 2}\right)\left(\dfrac{\ln 4}{\ln 3}\right)\left(\dfrac{\ln 5}{\ln 4}\right) \cdots \left(\dfrac{\ln 32}{\ln 31}\right) = \dfrac{\ln 32}{\ln 2} = \dfrac{\ln 2^5}{\ln 2} = \dfrac{5 \ln 2}{\ln 2} = 5$

12. (a)
$$\begin{aligned} f(-x) &= \ln\left(-x + \sqrt{(-x)^2 + 1}\right) = \ln\left(-x + \sqrt{x^2 + 1} \cdot \frac{-x - \sqrt{x^2+1}}{-x - \sqrt{x^2+1}}\right) \\ &= \ln\left(\frac{x^2 - (x^2 + 1)}{-x - \sqrt{x^2+1}}\right) = \ln\left(\frac{-1}{-x - \sqrt{x^2+1}}\right) = \ln\left(\frac{1}{x + \sqrt{x^2+1}}\right) \\ &= \ln 1 - \ln\left(x + \sqrt{x^2+1}\right) = -\ln\left(x + \sqrt{x^2 - 1}\right) = -f(x) \end{aligned}$$

(b) $y = \ln\left(x + \sqrt{x^2 + 1}\right)$. Interchanging x and y, we get $x = \ln\left(y + \sqrt{y^2+1}\right) \Rightarrow e^x = y + \sqrt{y^2+1} \Rightarrow$ $e^x - y = \sqrt{y^2 + 1} \Rightarrow e^{2x} - 2ye^x + y^2 = y^2 + 1 \Rightarrow e^{2x} - 1 = 2ye^x \Rightarrow$ $y = \dfrac{e^{2x} - 1}{2e^x} = f^{-1}(-x)$

13. $\ln(x^2 - 2x - 2) \leq 0 \Rightarrow x^2 - 2x - 2 \leq e^0 = 1 \Rightarrow x^2 - 2x - 3 \leq 0 \Rightarrow (x - 3)(x + 1) \leq 0 \Rightarrow$ $x \in [-1, 3]$. Since the argument must be positive, $x^2 - 2x - 2 > 0 \Rightarrow \left[x - \left(1 - \sqrt{3}\right)\right]\left[x - \left(1 + \sqrt{3}\right)\right] > 0$ $\Rightarrow x \in \left(-\infty, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, \infty\right)$. The intersection of these intervals is $\left[-1, 1 - \sqrt{3}\right) \cup \left(1 + \sqrt{3}, 3\right]$.

14. Assume that $\log_2 5$ is rational. Then $\log_2 5 = m/n$ for natural numbers m and n. Changing to exponential form gives us $2^{m/n} = 5$ and then raising both sides to the nth power gives $2^m = 5^n$. But 2^m is even and 5^n is odd. We have arrived at a contradiction, so we conclude that our hypothesis, that $\log_2 5$ is rational, is false. Thus, $\log_2 5$ is irrational.

15. Let d be the distance traveled on each half of the trip. Let t_1 and t_2 be the times taken for the first and second halves of the trip.

For the first half of the trip we have $t_1 = d/30$ and for the second half we have $t_2 = d/60$. Thus, the average speed for the entire trip is $\dfrac{\text{total distance}}{\text{total time}} = \dfrac{2d}{t_1 + t_2} = \dfrac{2d}{\dfrac{d}{30} + \dfrac{d}{60}} \cdot \dfrac{60}{60} = \dfrac{120d}{2d + d} = \dfrac{120d}{3d} = 40$. The average speed for the entire trip is 40 mi/h.

16. Let $f = \sin$, $g = x$, and $h = x$. Then the left-hand side of the equation is $f \circ (g + h) = \sin(x + x) = \sin 2x = 2 \sin x \cos x$; and the right-hand side is $f \circ g + f \circ h = \sin x + \sin x = 2 \sin x$. The two sides are not equal, so the given statement is false.

17. Let S_n be the statement that $7^n - 1$ is divisible by 6.

- S_1 is true because $7^1 - 1 = 6$ is divisible by 6.
- Assume S_k is true, that is, $7^k - 1$ is divisible by 6. In other words, $7^k - 1 = 6m$ for some positive integer m. Then $7^{k+1} - 1 = 7^k \cdot 7 - 1 = (6m + 1) \cdot 7 - 1 = 42m + 6 = 6(7m + 1)$, which is divisible by 6, so S_{k+1} is true.
- Therefore, by mathematical induction, $7^n - 1$ is divisible by 6 for every positive integer n.

18. Let S_n be the statement that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

- S_1 is true because $[2(1) - 1] = 1 = 1^2$.
- Assume S_k is true, that is, $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Then

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + [(2k + 1) - 1] &= 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) \\ &= k^2 + (2k + 1) = (k + 1)^2 \end{aligned}$$

  which shows that S_{k+1} is true.
- Therefore, by mathematical induction, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every positive integer n.

19. $f_0(x) = x^2$ and $f_{n+1}(x) = f_0(f_n(x))$ for $n = 0, 1, 2, \ldots$.

$f_1(x) = f_0(f_0(x)) = f_0(x^2) = (x^2)^2 = x^4$, $f_2(x) = f_0(f_1(x)) = f_0(x^4) = (x^4)^2 = x^8$,

$f_3(x) = f_0(f_2(x)) = f_0(x^8) = (x^8)^2 = x^{16}, \ldots$. Thus, a general formula is $f_n(x) = x^{2^{n+1}}$.

20. (a) $f_0(x) = 1/(2-x)$ and $f_{n+1} = f_0 \circ f_n$ for $n = 0, 1, 2, \ldots$.

$$f_1(x) = f_0\left(\frac{1}{2-x}\right) = \frac{1}{2 - \dfrac{1}{2-x}} = \frac{2-x}{2(2-x)-1} = \frac{2-x}{3-2x},$$

$$f_2(x) = f_0\left(\frac{2-x}{3-2x}\right) = \frac{1}{2 - \dfrac{2-x}{3-2x}} = \frac{3-2x}{2(3-2x)-(2-x)} = \frac{3-2x}{4-3x},$$

$$f_3(x) = f_0\left(\frac{3-2x}{4-3x}\right) = \frac{1}{2 - \dfrac{3-2x}{4-3x}} = \frac{4-3x}{2(4-3x)-(3-2x)} = \frac{4-3x}{5-4x}, \ldots$$

Thus, we conjecture that the general formula is $f_n(x) = \dfrac{n+1-nx}{n+2-(n+1)x}$.

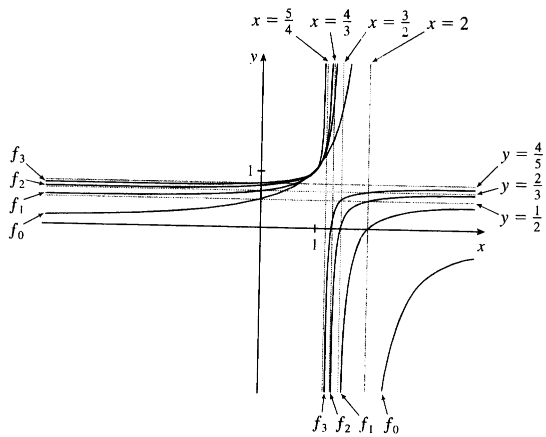
To prove this, we use the Principle of Mathematical Induction. We have already verified that f_n is true for $n = 1$. Assume that the formula is true for $n = k$; that is, $f_k(x) = \dfrac{k+1-kx}{k+2-(k+1)x}$. Then

$$f_{k+1}(x) = (f_0 \circ f_k)(x) = f_0(f_k(x)) = f_0\left(\frac{k+1-kx}{k+2-(k+1)x}\right) = \frac{1}{2 - \dfrac{k+1-kx}{k+2-(k+1)x}}$$

$$= \frac{k+2-(k+1)x}{2[k+2-(k+1)x]-(k+1-kx)} = \frac{k+2-(k+1)x}{k+3-(k+2)x}$$

This shows that the formula for f_n is true for $n = k+1$. Therefore, by mathematical induction, the formula is true for all positive integers n.

(b)



From the graph, we can make several observations:

- The values at $x = a$ keep increasing as k increases.
- The vertical asymptote gets closer to $x = 1$ as k increases.
- The horizontal asymptote gets closer to $y = 1$ as k increases.
- The x-intercept for f_{k+1} is the value of the vertical asymptote for f_k.
- The y-intercept for f_k is the value of the horizontal asymptote for f_{k+1}.